Dr Hendel's FM Lecture Notes

(c) Dr Hendel, 2017 (v3-1), 2022(v4-5.3)

TABLE OF CONTENTS

INTRODUCTION and PREFACE

Version 4.1 of the "Lecture Notes" has the same commitment as the previous versions: *A person mastering these notes and the problems associated with them will have a firm background in the Theory of Interest sufficient to pass the jointly administered Casualty Actuarial Society (CAS) and Society of Actuary (SOA) Financial Mathematics exam.* The following bulleted list contains new (and old) features of these notes.

- **Accuracy**: The notes have been used for several semesters. I wish to thank my students for catching many typos. I believe that almost all mathematical and English errors have been caught, enhancing book usability.

- **Pedagogically Challenging**: The book's pedagogic style is based on Deborah Hughes-Hallet's *rule of 4:* Material, both text and problems, are presented using four modalities: verbal, computational, geometric (time line) and formal. Such a format addresses executive function and corresponds to the challenging levels of the Marzano and Bloom-Anderson taxonomies of pedagogic challenge. I have published a more general treatment of pedagogy, unifying the hierarchies of Bloom, Anderson, Gagne, Marzano, Van-Hiele and others. It may be found in Chapter 11, "Leadership for Improving Student Success Through Higher Cognitive Instruction," of *Comprehensive Problem Solving and Skill Development for Next Generation Leaders,* R. Styon and J. Styron Eds, IGI publishing 2018. The book emphasizes that certain problems are more easily solvable using one modality versus another (For example, refinancing problems can be done quickly and efficiently using TV Calculator lines). I require my students to purchase the BA II Plus and these notes reflect that requirement.

- **One semester coverage, Lean and Lively Look:** The book is 133 pages and 19 chapters. The author covered the material at Towson University at about a chapter per class day in a 15 week, 2 days @ week course (Certain chapters had two days devoted to them.) At Towson this is a 4-unit course with 150 minutes per lecture day. Currently the course material is both undergraduate and graduate.

- **Additional Material:** Version 4-1 has some added material such as Chapter 3 which provides good problems to illustrate the money growth methods.

- **Problems**: The book uses real SOA problems with exam difficulty. The problems come from the "sample Financial Mathematics" problems published by the Society of Actuaries. There are currently (Aug 2019) 202. Prior to June 2017 there were 157. These problems are grouped at chapter ends. They are an integral part of the material. Solutions may be found on the SOA website. I started a project posting solutions to all problems using the rule of 4 but have not yet completed it nor have I removed typos.

- **Dedicated Websites**: The book website:
  www.Rashiyomi.com/math/DrHendelsFMLectureNotes.pdf
  The problem solution website is **www.Rashiyomi.com/math**. As already indicated the book is almost error free. It is not intended to update this significantly.

- **Abbreviations: QIT** stands for *Questions in Interest Theory*. The URL is given at each chapter end.

- **Communications:** I have benefitted from suggestions from both students and colleagues. Please feel free to write me at RHendel@Towson.Edu.

CHAPTER 1
COMPOUND INTEREST GROWTH

1.1 Money Growth: We are all familiar with the operations of a bank. Suppose the bank gives you 10% interest annually on your deposits. Why does the bank give you money? Because it invests your money in mortgages, bonds, and other investments. The bank pays you for your donation of the amount you deposited in the bank.

The amount you deposit is called the *principle.* The amount the bank pays you each year for giving your money to the bank to use, is called *interest.*

Example 1.10: If you deposit $1 *principle* on Jan 1 2017, then a year later the bank will pay you 10% x 1 = 0.10, or 10 cents *interest.* You *accumulated* value in the bank is the sum of principle and interest (minus withdrawals). In this case if you took nothing out you would have $1 + .10 = 1.10.

What happens at the end of the 2^{nd} year. The bank pays you 10% on the total 1.10 you have in the account at time t=1. So, the bank pays you 10% x 1.10 = 11 cents *interest*. So your total *accumulated* amount at t=2 is $1.21=$1.10+0.11.

Notice that the value of the dollar is a function of time. The *accumulated* value of the dollar at time 0 is, $A(0)= 1$. Similarly, $A(1)=1.1$, $A(2)=1.21$.

1.2 Timelines: We can summarize this with the following timeline. Timelines are basic tools in this course. The important features of timelines are the *t*-axis of time and the value of A(t). You can think of $A(t)$ as the *amount* function, it represents the *amount* in the bank at time *t.* The *A* in $A(t)$ also stands for *accumulation* and represents how much has *accumulated* in the account at time *t.* You can also think of a timeline as a graph with *x-axis* equaling *t*, and the *y*-axis equaling A(t).

Absolute Date	1/1/17	1/1/18	1/1/19		
Relative Date	0	1	2	…	t
A(t)	1	1.10	1.21	…	
Formulae	$A(0) = 1$	$A(1) = 1.1A(0)$	$A(2) = 1.1A(1) = 1.21A(0)$		$A(t) = A(0)\ (1+i)^t$

Table 1: *Illustration of a timeline with double time labels.*

1.3 Accumulation Formula: The following single formula governs all compound interest.

$$(1.1) \qquad A(t) = A(0)(1+i)^t$$

The proof of (1.1) is instructive since it gives insight into the meaning of money growth. We give a proof by induction.

Clearly the equation is true when $t=0$. It is also true when $t=1$. Indeed, the person deposited $A(0)$ at $t=0$. A year later the bank deposits $i*A(0)$ into the account. So, the person has $A(0)+i\,A(0) = (1+i)\,A(0)$ in the account.

Suppose, using an induction assumption, that the amount in the account at time t is $A(t)=A(0)\,(1+i)^t$. Then

The amount at time t+1 = *principle* at t + *interest* given by bank at t+1 $= A(t) + i(A(t))$ $= A(0)(1+i)^t + i\,[A(0)(1+i)^t]$ $= (1+i)[A(0)(1+i)^t]$ $= A(0)\,(1+i)^{t+1}$

Table 2: Verbal derivation of the compound interest formula, (1.1)

Notice how this derivation starts with an English sentence and then proceeds to fill in the mathematical formulae. In actuarial mathematics this is called a *verbal derivation.* Verbal derivations are very important in Actuarial Mathematics as they facilitate understanding and remembering formulae. This is unlike Calculus where verbal derivations might be frowned upon.

Equation (1.1) is the basic equation for growth of money by compound interest. There are four variables in the equation with the following meaning:

Symbol	Meaning
$A(t)$	Accumulated value in an account at time t
$A(0)$	Accumulated value at time 0
i	The interest rate, usually expressed as a percentage, that the bank pays yearly on the amount in the account at the beginning of the year
t	The number of years that has elapsed since the initial deposit.

Table 3: *Meaning of variables in accumulation function under compound growth.*

1.4 Compound Interest Problems: Table 3 gives rise to four distinct problems and four equation-solving techniques depending on the unknown.

What is unknown	Solving for the unknown in the equation $A(t)=A(0)(1+i)^t$
A(t)	Plug in other three values
A(0)	Divide $A(t) / (1+i)^t$ and plug
i	Divide both sides by A(0) and take the *t-th* root
t	Take logarithms of both sides, plug in, and solve for t.

Table 4*: Four possible problems associated with equation (1.1).*

We illustrate Table 4 by considering solving for the unknown t in the equation $1210=1000(1.1)^t$.

Following the advice in Table 4 we take logarithms and solve for t.

$$\begin{aligned} log(1210) &= log(1000(1.1)^t) \quad = log(1000) + tlog(1.1) \\ 7.0984 &= 6.9078 + 0.0953t \\ 0.1906 &= 0.0953t \\ 2 &= \frac{0.1906}{0.0953} = t \end{aligned}$$

1.5 The Time Value Calculator Line: A remarkable feature of the BA-II plus calculator is that you can solve all 4 problems using the *timevalue* (TV)line and hitting 4 buttons. This is summarized in Table 5. We use the formula $1210=A(2)=1000 \times 1.1^2$ to illustrate solving for each possible unknown.

$1210=1000 \times 1.1^{2.}$ What is unknown?	N	I	PV	PMT	FV
A(t)	2	10	-1000	0	CPT
A(0)	2	10	CPT	0	1210
i	2	CPT	-1000	0	1210
t	CPT	10	-1000	0	1210

Table 5: *Solution to the 4 problems in Table 4 using the TV line in the BA II plus.*

1.6 The Discount Factor, *v*: Prior to approaching problems we introduce one piece of notation used throughout the course. The symbol introduced, *v,* is called the *discount factor.* Notice how (1.3) and (1.1) say the same thing. However, (1.3) may be more useful since all cash flows can be conveniently evaluated at 0.

(1.2) $v = \dfrac{1}{1+i}$

(1.3) $A(0) = A(t)v^t$

How do you think of *v*? *v* is called the discount factor; v^t represents how much you must deposit at time t=0 to accumulate $1 at time *t.*

1.7 Problems: There is only one formula in this section. So, the problems test organizational capacity. Each problem involves several applications of (1.1). Knowing the formula is important. However, equally important, is knowing how to break a problem up into several problems each of which is governed by the formulas you know. The following problems are illustrative of such analysis.

Problem #1) Deposits of 100, 300, and 600 are made at times *t=1,3,6* respectively into an account earning 1.5%. Find the equivalent interest rate,*i*, under which 1000 deposited at time *t*=0 would equal the total accumulated value in this account at time *t*=6. **Hint**: We break this problem into 4 parts (100, 300, 600, and 1000)

Problem #2) (a) An initial deposit of 1000 doubles in 5 years at interest rate *i.*
(b) An initial deposit of *750* would accumulate to the same amount as in (a) in *n* years at rate *i-1*.
(c) Calculate the initial deposit needed to accumulate to 2500 at time 2*n* at rate 2*i.*
(**Hint**: We break this problem into three parts, a,b,c, each of which is governed by one TV line.)

1.8 Solutions: Again, we emphasize, there is only one formula, (1.1) or (1.3). So, solutions introduce key *organizational* techniques.

Solution to Problem #1) The following timeline summarizes key points in the solution. Notice that this timeline has 5 sub timelines. Use of graphical methods is a fundamental technique.

t	0	1	2	3	4	5	6
$100 account		100					$A(6)=100 \times 1.015^5=$ 107.73
$300 account				300			$A(6)=300 \times 1.015^3=$ 313.71
$600 account							$A(6)=600$
Total of first account I							Sum: 107.73+313.71+600= 1021.44
$1000 account II	1000						$A(6)=1000 \times (1+i)^6$

The problem solution can be completed by writing an *equation of value* equating the accumulated value of the first and 2nd account at time t =6. The equation of value can be solved algebraically or computationally.

EOV: Equation of value: $A_I(6) = A_{II}(6)$

$$1021.44 = 1000(1+i)^6$$

The solution using the TV line is as follows:

N	I	PV	PMT	FV
6	CPT=0.3542%	-1000	0	1021.44

Solution to Problem #2) This problem can be solved using 3 TV lines with associated EOV.

N	I	PV	PMT	FV	EOV

5	CPT=14.87%	-1000	0	2000	$2000=1000 \times (1+i)^5$
CPT=7.55	*i-1*	-750	0	2000	$2000=750 \times (.99+i)^n$
2n	2i	CPT=49	0	2500	$2500=X(1+2i)^{2n}$

1.9 Summary of Concepts and Techniques:

1) Principle, 2) Interest, 3) Accumulated amount A(t), 4) *Compound interest formula, 5) Discount factor, v, 6) Timelines, (7)Verbal derivations), 8) TV lines, 9) 4 methods of solving equations,10) breaking up problems.*

1.10 SOA Problems: #20 (EOV),23 (Inflow-Outflow), 32 (NPV),
In the following problems, nominal rates (Chapter 2) and Bonds are foreshadowed:
94 (quadratic approach), 111 (Resembles somewhat problem 3 in sections 1.7 and 1.8)

Source of SOA problems: https://www.soa.org/Files/Edu/2015/edu-2015-exam-fm-ques-theory.pdf
Source of my solutions: www.Rashiyomi.com/math/

CHAPTER 2

NINE ACCUMULATION FUNCTIONS

2.1 Default Money Growth Method: In the previous chapter, Chapter 1, we explored the use of compound interest as a means of accumulating money. Compound interest is the general method used. If no accumulation function is stated, you assume the method is compound interest growth. Various English terms indicate compound interest: *effective rate, compounded annually, annual rate.*

2.2 Other Money Growth Methods: In this chapter we review 9 methods by which money accumulates. They are compactly summarized in Table 2.1 along with their English descriptions and standard symbols. (The reasons these are sometimes used vs. annual compound are not discussed in any great detail).

Name of method	English description	Special symbols	Formula	Formula for A(E) from deposit at B, of $A(B)$
Compound interest	*effective, compounded annually, annual rate*	i	$A(t) = A(0)(1+i)^t$	$A(E) = A(B)(1+i)^{(E-B)}$
Simple interest	Simple interest	i	$A(t) = A(0)(1+it)$	$A(E) = A(B)(1+i(E-B))$
Nominal Compound Interest	*compounded m-th-ly, compounded m times a year, payable m-th-ly, nominal rate i payable m times a year*	$i^{(m)}$ Note: $i^{(m)}/m$ is the rate per m-th of a year	$A(t) = A(0)\left(1+\frac{i^{(m)}}{m}\right)^{mt}$	$A(E) = A(B)\left(1+\frac{i^{(m)}}{m}\right)^{m(E-B)}$
Simple discount	Simple discount	d	$A(t) = \frac{A(0)}{1-dt}$	$A(E) = \frac{A(B)}{1-d(E-B)}$

Nominal discount	*discount compounded m-thly, discount compounded m times a year, nominal discount rate compounded m times a year, nominal discount d payable m times a year*	$d^{(m)}$ Note: $d^{(m)}/m$ is the discount rate per m-th of a year	$A(t) = \frac{A(0)}{\left(1 - \frac{d^{(m)}}{m}\right)^{mt}}$	$A(E) = \frac{A(B)}{\left(1 - \frac{d^{(m)}}{m}\right)^{m(E-B)}}$
Compound discount	*Discount compounded annually, annual discount rate, effective annual discount rate*	d	$A(t) = \frac{A(0)}{(1-d)^t}$	$A(E) = \frac{A(B)}{(1-d)^{(E-B)}}$
Force	*force of interest*	δ_t	$A(t) = A(0)e^{\int_0^t \delta_s ds}$	$A(E) = A(B)e^{\int_B^E \delta_s ds}$
Constant Force	*force of interest*	δ	$A(t) = A(0)e^{\delta t}$	$A(E) = A(B)e^{\delta(E-B)}$
General accumulation function		*a(t)*	$A(t) = A(0)a(t)$	Use 2 equations $A(B) = A(0)a(B)$,solve for $A(0)$ $A(E) = A(0)a(E)$, solve for $A(E)$

Table 2.1: Description of 9 types of accumulation functions.

2.2 Concepts Related to Money Growth: Certain functions and concepts are related to all accumulation functions. They are summarized in Table 2.2.

Concept	Formula	Typical usage
Interest Amount	$I_{B,E} = A(E) - A(B)$	See next two rows
Period interest rate	$i = \frac{I}{A(B)}$	$A(E) = A(B)(1+i)$
Period discount rate	$d = \frac{I}{A(E)}$	$A(E) = \frac{A(B)}{1-d}$
Period interest factor	$(1+i)$	$A(E) = A(B) \times \text{Interest factor}$
Period discount factor	$v(t) = \frac{1}{a(t)}$, with $a(t) = \frac{A(t)}{A(0)}$	Present value of $A(t)$ at t=0 $= v(t)A(t) = \frac{A(t)}{a(t)}$
PV, Present Value	*The present value of X at time t is X/a(t). That is, an amount of X/a(t) at time t=0 would enable purchase of X at time t* *If X is at time t is equivalent to Y at time s then Y v(s) = X v(t) or Y = X v(t)/v(s) where v(u) = 1/a(u).*	WARNING ☹ For compound interest and simple interest you can use relative time; you can use v^{t-s} from time *s* to *t* For other problems you always go back to 0. See last row of Table 2.1 for accumulation

Table 2.2: Several concepts relevant to *all* accumulation functions. Note B and E refer to period beginning and end.

2.3 Concepts Associated with Problems: We close the chapter by mentioning five very important concepts.

- Equation of Value (EOV): The Equation of value is the *key* to solving any problem with multiple parts. The EOV relates the accumulation functions of various timelines to each other. Very often, *simply stating the EOV using high level descriptions* is enough of a hint to solve the problem. Note that the accumulation function associated with one timeline is not an EOV; an EOV relates the accumulation function from all timelines in the table.
- English EOV: It is not possible to list all course formulae! The reason for this is that the EOV may be general and high level. This course emphasizes English EOV. An example of an English EOV is *Inflow = Outflow*. This last equation is an English EOV that shows how to deal with a problem with many payments and deposits. The deposits are *inflows* while the payments are *outflows*. The EOV for the problem asserts that the PV of all the *inflows*

equals the PV of all the *outflows*. You then have to decide *in each problem* what the *inflows and outflows* are.
Another example of an English EOV is *Interest Amount = Interest Amount.* This English EOV may also be expressed as $I^{(A)}{}_{7.5,8} = I^{(B)}{}_{15,16}$, in words, the amount of interest accumulated in Account A between period times 7.5 and 8 is equal in amount to the amount of interest accumulated in account B between period times 15 and 16. Why do we need this English EOV? Because there are 9 possibilities for accumulation functions for the A account and 9 possibilities for the accumulation for the B account. This gives rise to several dozen equations for the equality of interests. You can't write all these down; you can't memorize them. However, you can start with the English EOV and then *develop* the algebraic EOV for that particular problem. This is a fundamental problem-solving technique.

- Interest rate for half a year: What is the interest rate for half a year under nominal interest compounded twice a year vs. under annual compound interest? Here is a different way to formulate this problem. If I deposit 1 at time t=0, what is the value of $A(1/2)$ under a) compound interest b) nominal interest (twice a year). Assume a 10% value for the compound and nominal rate.
Nominal vs Compound: Under nominal interest compounded twice a year, the rate per half year is 10%/2 = 5%. So $A(1/2)$=1.05. However, we don't immediately know the rate under compound effective interest. Assume that the rate is j. That is, assume that $1+j = A(1/2)$. Then t=1 corresponds to the 2nd half year (t=2). So $A(1) = 1(1+j)^2$. But $A(1) = 1.1$. So, we obtain j by solving $(1+j)^2 = 1.1$, and hence $j = 4.88\%$.

- Computing Force vs Accumulation: Table 2.1 computes accumulation given force. Suppose you wanted to go in the opposite direction: Given accumulation, compute force. Inspecting Table 2.1, we see that we must take logarithms and differentiate to get the function under the integral sign, the integrand. This gives:
(2.1) $\delta_t = \dfrac{\frac{dA}{dt}}{A(t)} = \dfrac{d \log A(t)}{dt} = \dfrac{A'(t)}{A(t)}$

The following tip about problem difficulty is worthwhile to note. Equation (2.1) contrasts with the force equation in Table 2.1. This is a situation where we have *two* formulas connected with one concept, *force.* Whenever a 2:1 formula to concept ratio exists, problem difficulty goes up. Why? Because the student can't just plug in; they must first decide which formula to use. In this case, it is easy:

  - Are you given accumulation and want the force? Then use (2.1)
  - Are you given force and want the accumulation? Then use Table 2.1

- Actuarial Equivalence: This is a very important concept which dominates both this course (in all EOVs) and the next course MLC. *Two money growth functions are actuarially equivalent if the ending amount at time t=E is the same for both functions whenever the beginning amount at time t=B is the same.* A typical application might be the following:

Find the effective rate, i, actuarially equivalent to an account subject to several money growth forces during the same period.

- Technique for analyzing a multi-step problem: A problem can have multiple i) rates, ii) money growth methods, iii) amounts. The fundamental technique for analyzing a multi-step problem into component problems is to break it up into subproblems with each problem having i) one amount, ii)one rate, and iii) one money growth method. You then label that particular timeline with a beginning *B*, an end, *E*, a beginning amount at *B*, and an end amount reflecting the amount at time *B*, and the particular money growth method used. You relate the various subproblems by using appropriate equations of values.
- Solving multiple EOVS: If you have
  - Two equations, with one having one unknown and one having two unknowns, then
    - Solve the one-unknown problem first and
    - Plug the result into the 2-unknown question.
  - Two equations, with both having two unknowns, then
    - Solve for one variable in terms of the other variable (for example solve x = *some function of* y); then
    - Plug this function of y into the other equation and you will have one equation in one unknown, so you can solve for y. Finally,
    - Reuse either equation, plug in the value of y and obtain the value of x.

2.4 SOA Problems:

Source of SOA problems: https://www.soa.org/Files/Edu/2015/edu-2015-exam-fm-ques-theory.pdf
Source of my solutions: www.Rashiyomi.com/math/

FROM QIT: (So 1 refers to QIT Problem #1)

1 Force - nominal (Good comparison problem)

3 (Interest amount-nominal-simple),

12 (Piecewise functions),

13 (Force-interest amount),

21(Difficult problem, not recommended by me: Continuous payments),

27(Interest amount),

61 (Force – Integration practice, substitution),

77 (Nominal-nominal),

79 (Compound-Force),

95 (Nominal-Nominal; Discount-Interest),

104 (Difficult problem; not recommended by me),

105 (Piecewise: Compound-Force),

135(Force-Simple).

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M01#49 -Force-nominal-simple (Comparison problem/Abstract)

M00#37 - Force-nominal - Piecewise functions

N01#1 - Force - Interest Amount (Algebra)

N00#53 - Force - compound (Comparison / Algebra)

M01#45 - Force - nominal discount (Good comparison problem)

CHAPTER 3
Money Growth: Advanced Tips

3.1 Money Growth: Chapters 1 and 2 covered the basic principles of Money Growth. In this chapter we go over advanced concepts and subtleties. Mastery of the problems in this chapter will equip the student with what is necessary to answer any type of money growth question. Another way to think of this chapter is that it gives problems where there are typical foils to avoid

3.2 Topics Covered in this Chapter: The problems in this chapter illustrate the following

Section	Title	What is Dealt With
3.3	QIT #12	Problems with multiple deposits, multiple rates, and multiple money-growth functions
3.4	QIT #135	Calculation force at point of time
3.5	QIT #3	Questions on *I*, Interest Amount
3.6	Problem	Undetermined Coefficient Method, Pricing not at 0, Quadratic $a(t)$
3.7	Problem	Calculating interest rate for a specific year

Section 3.3: QIT #12, Multiple Deposits, Rates, Money Growth functions

Jeff deposits 10 into a fund today and 20 fifteen years later. Interest for the first 10 years is credited at a nominal discount rate of d compounded quarterly, and thereafter at a nominal interest rate of 6% compounded semiannually. The accumulated balance in the fund at the end of 30 years is 100.

Calculate d

PROBLEM SUBTLETY: Multiple deposits, rates, and money growth functions

PROBLEM APPROACH: Break up into units. A unit means, *can be solved by one formula.*

NUMBER PROBLEMS: 3 Problems

Problem Number	Amount	From t=, To t=	Rate
1	10	0 to 10	*d, quarterly*
2	$A_{10}(10)$	10 to 30	6% *semi-annually*

3	15	15 to 30	6% *semi-annually*

TIMELINES:

t=0 *t=10*

----

10 $10/(1-d/4)^{40}$

t=10 *t=30*

$A_{10}(10)$ $A_{10}(10)\ (1+6\%/2)^{2x20}$

t=15 *t=30*

----

20 $20\ (1+6\%/2)^{2x15}$

Why does this work so easily? Because each of the three units *can be solve by one formula*

EOV: Equation of Value (Relates all parts of problem)

English version: *Inflow* = *Outflow*

Accumulated value of 10 deposit + Accumulated value of 15 deposit = 100

$$[10(1-d/4)^{-4x10}]\ (1+6\%/2)^{2x20} + 20\ (1+6\%/2)^{2x15} = 100$$

ALGEBRA:

Solve for *d*. 1 equation in 1 unknown to a power. Need to take roots to get *d* by itself.

SOLUTION: *d* =4.5318%

Section 3.4: QIT #135, Evaluating Force at a Point

At time 0, Cheryl deposits *X* into a bank account that credits interest at an annual effective rate of 7%. At time 3, Gomer deposits 1000 into a different bank account that credits simple interest at an annual rate of *y*%. At time 5, the annual forces of interest on the two accounts are equal, and Gomer's account has accumulated to *Z*.

Calculate *Z*.

PROBLEM SUBTLETY: Calculating force of interest at a point in time.

PROBLEM APPROACH: Calculate $A(t)$. *Then* d/dt log $A(t)$. *Only then* plug in t=5.

NUMBER PROBLEMS: 2 problems: Cheryl and Gomer

TIMELINES:

0 5

X $A(t) = X(1.07)^t$ ← ☹ *Do not plug in 5 yet!!!*

3 *5*

1000 $Z = 1000(1+(t-3)y\%)$ ← *t-3, not t; Do not plug in 5 yet!!!*

EOV: Equation of Value (Relates all parts of problem)

$$\delta^{(Cheryl)}_t \big|_{t=5} = \delta^{(Gomer)}_t \big|_{t=5}$$

ALGEBRA: $\frac{d}{dt} \log A(t)\big|_{t=5} = A'(t) / A(t) \big|_{t=5}$

$A(t) = X(1.07)^t$ → $\log A(t) = \log(X) + t \log(1.07)$ → $\delta_t = d/dt \log A(t) = 0 + \log(1.07)$

$A(t) = 1000(1+(t-3)y\%)$ → $A'(t) = 1000\, y\%$ → $\delta_t = A'(t)/A(t) = y\% / (1+(t-3)y\%)$

Set these 2 equal at t=5: $\log(1.07) = y\% / (1+2y\%)$ → Linear equation→ y = 7.82

SOLUTION: $Z = 1000(1+(t-3)y\%) = 1000(1+(5-3) \times 7.82\%) = \$1156.49$

Section 3.5: QIT #3 (Difficulty 4), Questions on I, Interest Amount

Eric deposits 100 into a savings account at time 0, which pays interest at an annual nominal rate of i, compounded semiannually.

Mike deposits 200 into a different savings account at time 0, which pays simple interest at an annual rate of i.
Eric and Mike earn the same amount of interest during the last 6 months of the 8^{th} year.

Calculate i.

PROBLEM SUBTLETY: Question on I vs A

PROBLEM APPROACH: Skillful use of EOV: 1) High level and short, 2) multi-step

NUMBER PROBLEMS: 2 Problems, Eric, Mike

TIMELINES:

t=0	*t=7.5*	*t=8*
100	$100(1 + i/2)^{15}$	$100(1 + i/2)^{16}$

t=0	*t=7.5*	*t=8*
200	$200(1+7.5 \times i)$	$200(1+8 \times i)$

Equation of Value (Relates all parts of problem)

English: Same Interest (I) = Same Interest (II) (Last 6 months of 8th year)

$I^{(I)}_{7.5,8} = I^{(II)}_{7.5,8}$ ← *Very important: High Level Brief (no formulae)*

$A^{(I)}_8 – A^{(I)}_{7.5} = A^{(II)}_8 - A^{(II)}_{7.5}$ ← *Important: Definition of I; no algebra yet*

ALGEBRA:

$A^{(II)}_8 - A^{(II)}_{7.5} = 200(1+8i) – 200(1+7.5i) = 0.5i \times 200 = 100i$

$A^{(I)}_8 – A^{(I)}_{7.5} = 100(1+i/2)^{16} – 100(1+i/2)^{15}$

$= 100(1+i/2)^{15} (1+i/2 - 1)$, ← Factoring

$= i/2 \times 100(1+i/2)^{15} = 50i(1+i/2)^{15}$

Set above 2 equal (EOV): $100i = 50i (1+i/2)^{15}$ → $2 = (1+i/2)^{15}$

$2^{1/15} = 1 + i/2$

$i = 2(2^{1/15} - 1)$

SOLUTION: $i = 9.46\%$

About Calculus: You don't have to remember all of Calculus for this course. Here is what you should know

- Derivatives of x^n, e^x, $\ln(x)$
- Integrals of x^n, e^x
- Sum rule, product rule, chain rule, substitution technique
- Tayler's formula $f(x_0 + \text{little bit}) \sim f(x_0) + f'(x_0)(\text{little bit}) + f''(x_0) (\text{little bit})^2/2+\ldots$

Section 3.6: PROBLEM: Undetermined coefficients, P not at 0, *v*(t)

An account is governed by a quadratic accumulation function.

The cost of a dollar at t=1 is 50 cents and the cost of a dollar at 2 is 20 cents.

A special package to be sold by you at t=2 for P. In exchange for paying you P, the person buying it will receive \$100 at t=3 and \$200 at t=4.

Calculate the price, P, of this special package using rates at t=0.

PROBLEM SUBTLETY: i) *Don't know A(t) form;* ii) *Pricing not at 0*; iii) *v(t) not compound*

PROBLEM APPROACH: i)Method of undetermined coefficients;ii) Discount to 0; iii)v(t)=1/a(t)

NUMBER PROBLEMS: 4 problems: i) Find A, ii) 200, iii) 100, iv) P

TIMELINES:

Using 1/a(t) = v(t)   0   t
a(t) = A(t)/A(0)   ---
                              v(t)=1/a(t) A(t) = $v(t)\ a(t)$ = 1/a(t) x a(t) = 1

0	2	3	4
P $v(t)$	P		
100 $v(t)$		100	
200 $v(t)$			200

Equation of Value (Relates all parts of problem)

English: PV(Price-outflow) = PV (Benefits-inflow)

$$P\, v(2) = 100\, v(3) + 200\, v(4)$$

$$P / a(2) = 100/a(3) + 200/a(4)$$

ALGEBRA:

FIRST: Solve for a(t) – Method of Undetermined Coeffiients: $a(t) = At^2 + Bt + C$

$1 = a(0) = A\,0^2 + B0 + C$ ➔ C=1

$2 = 1/0.5 = a(1) = A\,1^2 + B + C$ ➔ A+ B = 1

$5 = 1/0.20 = a(2) = 4A + 2B + C$ ➔ C= 1; A =1, B=0

⇨ $a(t) = t^2 + 1$

SECOND: Solve EOV equation

$$P / a(2) = 100/a(3) + 200/a(4)$$

$P / 5 \quad = 100 / 10 \quad + 200 / 17$

SOLUTION: *P = $108.82*

SECTION 3.7: PROBLEM: *Determining Rates for a Particular Year*

An account is governed by simple rate of discount of 2% per year.

a) Calculate the interest rate for year 4

b) Calculate the annual effective rate for years 2 and 3

PROBLEM SUBTLETY: Requires integrating several formulas: i) interest rate, ii) interest amount, iii) Accumulated value, and iv) meaning of year X.

PROBLEM APPROACH: Go backwards (starting at end) through each of the necessary steps listed in *problem subtlety*.

iv) What is *year i*:

0-----------1------------2-----------3-----------4-------------------*i-1* -------- *i*

Year 1 Year 2 Year i

Year *i* ends at time *t=i*. Year *i* begins at time t=*i-1*.

iii) Accumulated Values: (Use formulas from Chapter 2)

A(1) = A(0)/(1-2%) = A(0)/98%=A(2) = A(0)/(1-2*2%) = A(0)/96%; A(3) = A(0)/94%; A(4) = A(0)/92%

ii) Interest amount: $I_{3,4}$ = A(4)-A(3) = A(0)/0.92-A(0)/0.94 = 0.0231 A(0) ; $I_{1,3}$ = A(3)-A(1) = 0.0434 A(0)

i)Solution to Problem a): $i_{3,4}$ = $I_{3,4}$/A(3) =0.0231 A(0) / (1.0638 A(0)) = 0.0231/1.0638 = 2.17%

Solution to Problem b): $i_{1,3}$ = $I_{1,3}$/A(2) = 0.0434 A(0)/[1.0204 A(0)] = 4.26%; Let *i* be the annual effective rate from 1 to 3

Then $(1+i)^2 = 1.0426 \Rightarrow i = 2.1\%$

CHAPTER 4
ANNUITIES

4.1 Annuities: An annuity refers to a periodically repeating payment. Table 4.1 presents the timeline for the simplest type of annuity: *n* end-of-year payments of 1.

Time	0	1	2…	n
Cashflows		1	1…	1
PV	$a_{\overline{n}\mid i}$	v	v^2…	v^n

Table 4.1: *Cashflows for an n-year annuity with end-of-year payments of 1.*

4.2 Formula for Present Value: The symbol for the *n*-year annuity with end-of-year payments of 1 is presented in Table 4.1. The use of *i* in this symbol is optional but very helpful. Using Table 4.1 we can infer the EOV.

$$(4.2)\ a_{\overline{n}\mid i} = v + v^2 + \ldots + v^n.$$

Many textbooks use the formula for sums of a geometric series to obtain a closed-form formula from (4.2). The following elegant verbal derivation is presented by Kellison. Verbal derivations, while frowned upon in Calculus, are praised in Actuarial mathematics and are a necessary tool for full comprehension.

Kellison's proof is summarized in Table 4.3.

Time	PV	0	1	2	…	n
Cashflows of depositor	$1-v^n$	1	0	0	…	-1
Cashflows of Bank	$ia_{\overline{n}\mid i}$		i	i	...	i

Table 4.3: *Kellison's proof of the formula for an annuity of 1 with end of year payments.*

The following narrative will explain Table 4.3. The depositor deposits 1 at time t=0. The bank then deposits *i* into his account at t=1. The depositor withdraws the *i*, but retains the 1. So, the bank, at time t=2, deposits i. The depositor withdraws the i. This continues until time $t=n$. The depositor then withdraws the 1 deposited at t=0.

The depositor has spent 1-v^n. Indeed, (s)he spent 1 at t=0 but only got back 1 at $t=n$, which has a present value of v^n. Contrastively, the bank has spent $ia_{\overline{n}|i}$. Indeed, it has spent periodically, an amount i, at the end of each period for n periods.

There are no funds left in the bank at $t=n$. So, depositor and bank must have spent equal amounts. We therefore obtain the fundamental EOV for a level-payment end-of-year annuity.

$$\textbf{(4.4)}\ ia_{\overline{n}|i} = 1 - v^n \longrightarrow a_{\overline{n}|i} = \frac{1-v^n}{i} = v + v^2 + \ldots + v^n$$

4.3 Calculator TimeValue Line: To compute numerical values, one does not need to use (4.4) since the TV line can compute it with a few strokes. Table 4.5 illustrates this fundamental technique.

N	I	PV	PMT	FV	Calculated
5	1	CPT=4.8534	-1	0	$a_{\overline{5}|1\%}$
10	2	CPT=179.65	-20	0	$20a_{\overline{10}|2\%}$
10	CPT=2	179.65	-20	0	Solve for i, $20a_{\overline{10}|i\%} = 179.65$
10	2	0	-20	CPT=218.99	FV of annuity. 1.02^{10} x 179.65

Table 4.5: *Illustrative calculations of annuity-related entities with the TV line.*

4.4 Six Level Annuity Functions: In the previous sections we have calculated the PV of an annuity. The PV is evaluated one period before the first payment. But there are three other approaches to evaluation. They are summarized in Table 4.6 along with explanatory names and symbols.

Time	10	11	12	...	10+n	11+n
Cashflow		1	1	…	1	
Annuity immediate	$a_{\overline{n}|i}$					
Annuity due		$\ddot{a}_{\overline{n}|i}$				
					$s_{\overline{n}|i}$	
						$\ddot{s}_{\overline{n}|i}$

Table 4.6: *Four annuity functions each associated with the same set of cashflows but evaluated at different points of time.*

It turns out there are 6 functions related to the basic annuity. They have in common that the payments are *level,* the same each period. They differ on i) whether payments are made at the beginning of the year or the end of the period, ii) at what point the evaluation takes place, and iii) whether payments are finite or infinite. Table 4.10 summarizes the symbols, names, formulae, TV lines and English descriptions of these 6 functions.

Perhaps we should clarify how the formulae are derived with some illustrative derivations.

Prove the formula for $s_{\overline{n}|i}$:

The accumulated value takes place *n* periods after 0. We can use the fundamental formula for compound interest, (1.2), relating a beginning and ending deposit.

$$\textbf{(4.7)}\ s_{\overline{n}|i} = (1+i)^n a_{\overline{n}|i} = (1+i)^n \times \frac{1-v^n}{i} = \frac{(1+i)^n - 1}{i}.$$

Proof for the formula for a perpetuity immediate:

We can take limits in an annuity immediate letting *n,* the number of years go to infinity. We obtain,

$$\textbf{(4.8)}\ a_{\overline{\infty}|i} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

Proof for the formula for annuity due:

A deposit of 1 at *t*=0 is actuarially equivalent to a deposit of 1+*i* at *t*=1. Similarly, a deposit of 1 at *t*=1 is actuarially equivalent to a deposit of 1+*i* at *t*=1. In general, each payment of 1 in the annuity due is equivalent to a payment of 1+*i* one year later. Hence, we obtain the following:

$$\textbf{(4.9)}\ \ddot{a}_{\overline{n}|i} = (1+i)a_{\overline{n}|i} = (1+i)\frac{1-v^n}{i} = \frac{1-v^n}{\frac{i}{1+i}} = \frac{1-v^n}{d}$$

Here we have used the fact that if *i* and *d* are equivalent then $1+i = (1-d)^{-1}$, implying $d = i/(1+i)$.

Symbol	Name	Formula	TV Line	English Description
$a_{\overline{n}|i}$	Annuity immediate (certain)	$\frac{1-v_i^n}{i}$	N: *n*; I: *i*; PV: CPT; PMT: -1; FV: 0	*PV one period before 1st payment of n end-of year payments of 1*

<table>
<tr><td>$s_{\overline{n}|i}$</td><td>No name</td><td>$\frac{(1+i)^n - 1}{i}$</td><td><table><tr><td>N</td><td>I</td><td>PV</td><td>PMT</td><td>FV</td></tr><tr><td>n</td><td>i</td><td>0</td><td>-1</td><td>CPT</td></tr></table></td><td>FV at time of last payment of n end-of-year payments of 1</td></tr>
<tr><td>$\ddot{a}_{\overline{n}|i}$</td><td>Annuity due (certain)</td><td>$\frac{1 - v^n}{d}$</td><td><table><tr><td>N</td><td>I</td><td>PV</td><td>PMT</td><td>FV</td></tr><tr><td>n</td><td>i</td><td>CPT</td><td>-1</td><td>0</td></tr></table>Enter BGN mode</td><td>PV at time of 1st payment of n beginning-of year payments of 1</td></tr>
<tr><td>$\ddot{s}_{\overline{n}|i}$</td><td>No name</td><td>$\frac{(1+i)^n - 1}{d}$</td><td><table><tr><td>N</td><td>I</td><td>PV</td><td>PMT</td><td>FV</td></tr><tr><td>n</td><td>i</td><td>0</td><td>-1</td><td>CPT</td></tr></table>Enter BGN mode</td><td>FV one period after last payment of n beginning of year payments of 1</td></tr>
<tr><td>$a_{\overline{\infty}|i}$</td><td>Perpetuity immediate</td><td>$\frac{1}{i}$</td><td><table><tr><td>N</td><td>I</td><td>PV</td><td>PMT</td><td>FV</td></tr><tr><td>10000</td><td>i</td><td>CPT</td><td>-1</td><td>0</td></tr></table></td><td>PV one period before 1st payment of an infinite number of end-of year payments of 1</td></tr>
<tr><td>$\ddot{a}_{\overline{\infty}|i}$</td><td>Perpetuity due</td><td>$\frac{1}{d}$</td><td><table><tr><td>N</td><td>I</td><td>PV</td><td>PMT</td><td>FV</td></tr><tr><td>10000</td><td>i</td><td>CPT</td><td>-1</td><td>0</td></tr></table>Enter BGN mode.</td><td>PV at time of first payment of an infinite number of begin-year payments</td></tr>
</table>

Table 4.10. *The six types of level annuities with their symbols, names, formulae, TV lines and English descriptions. The calculator evaluations of perpetuities are (very good) approximations.*

Before closing this section, we explain the terms *level* and *certain.* These are optional adjectives for descriptions of annuities. They mean as follows:

- A *level* annuity refers to the same payment each period. This contrasts with annuities that increase or decrease payments by a fixed amount each period as well as annuities that increase or decrease geometrically.
- A *certain* annuity refers to the fact that payments are certain. This contrasts with *life* annuities where payments are dependent on survival. The following illustrates this nicely: A lottery may be taken as an annuity certain since even if the person dies, the payments will continue to his or her estate. Contrastively, a life annuity has non-certain payments. The insurance company makes payments for as long as the owner is alive. Upon death, payments cease.

4.6 Rule of Three: Consider the payment scheme in Table 4.10. Notice its 3 distinct attributes:

- *Deferral* – The first payment is deferred to time $t = b+1$.
- Payment *Amount* – The payment may be p and not necessarily equal to 1.
- Payment *pattern* – it could be an annuity immediate, due, or increasing.

The identification of these three attributes leads to the PV of the payments. Actuaries refer to this as the *rule of three* since these three attributes determine PV.

Time		…*b*	*b+1*	*b+2*	…	*b+n*
Payments			*p*	*p*	…	*p*

Table 4.11: *n i) level payments of ii) p beginning at time iii) b+1. This table illustrates the rule of 3.*

The fundamental *rule of three,* (4.11), gives the following PV of this stream of payments at 0.

(4.12) $v^b p a_{\overline{n}|i} = v^{b+1} p \ddot{a}_{\overline{n}|i}$

4.8 Conversions: There are two periods connected with every annuity:

- *Payment period*: The period or distance in time between any two payments.
- *Interest period*: The period of time used to express the interest rate.

Formulae only work if the *interest period* and the *payment period* are the same.

Here are some examples.

Example 4.12: An annuity pays $5 at the end of every year at a nominal rate of 10% convertible quarterly.

Example 4.13: An annuity pays $5 at the end of every half a year and an annual effective rate of 10%.

Example 4.14: An annuity pays $5 at the end of every quarter of a year at a nominal rate of 10% convertible twice a year.

Solutions to examples: For each example we have to *convert* the interest rate into a rate per payment period. The three examples illustrate three typical situations.

Solution to Example 4.12: The interest rate is 2.5% a quarter. But we need an interest rate per year since the payment period is per year. So we must convert the 2.5% per quarter to an interest rate per year. Notice that we cannot use 4 x 2.5% = 10%. Here is a timeline illustrate the conversion technique.

0----------1/4-------------1/2----------------3/4---------------------------1

1----------1.025-----------1.025^2------------1.025^3----------------------1.025^4=1.1038

So, the problem is about a $5 end-of-year periodic deposit at an effective rate of 10.38% per year.

Solution to Example 4.13: Here the $5 is paid at the end of every half a year but the interest rate is 10% a year. We have to *solve* for the unknown rate per half a year, say *j*. Here is a timeline

```
0---------------------------1/2---------------------------------------------1
1--------------------------(1 + j)¯--------------------------------------(1+j)²=1.10
```

We can solve this equation and find that the problem is about a $5 end-of-quarter payment at a rate of 4.88% per quarter.

Solution to Example 4.14: This is tricky. We are given a payment period of a quarter and an interest period of half a year (10% nominal twice a year is 5% per half year). So we must convert the half year interest period to a quarter year rate. Here is the timeline

```
0----------1/4-------------1/2--------
1----------1+j-----------(1 + j)²=1.05
```

Solving $(1+j)^2 = 1.05$ we obtain $j = 2.4695\%$. So the problem is about a $5 end-of-quarter payment at 2.4695% per quarter.

4.8 SOA Problems: There are quite a few problems. The problems illustrate subtleties in the theory either hinted at above or not fully covered. The problems should be worked through until mastery. Next to each problem we have parenthetically indicated a major focus of the problem.

Source of SOA problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:
#2 (Difference of squares),
#17 (Deferrals),
#25 (Deferrals, Multiple EOV),

#29 (Conversions),
#48 (College Funding),
#49 (Definitions),
#96 (Definitions),

#97 (Conversions, comparison, 2 interest rates),
#98 (College funding),
#99 (2 interest rates),
#110 (Final payment),
#134(Annuity -Perpetuity),
#137 (Annuity – College Funding),
#140 (Annuity – Perpetuity),
#150 (Annuity – 2 Rates).

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M01#50 Level-Level-Perpetuity

M05#24 Level - Algebra

M00#14 Perpetuity - Perpetuity - Conversion

M03#33 Level - (Level+Deferral Factors

CHAPTER 5
INCREASING, DECREASING ANNUITIES

5.1 Annuities: The timelines for the *increasing annuity* and *decreasing annuity* are shown respectively in Tables 5.1 and 5.2. The reason for their names should be clear.

Time	0	1	2	3…	n
Cashflows		1	2	3…	n
PV	$(Ia)_{\overline{n}\|i}$	v	$2v^2$	$3v^3$…	nv^n

Table 5.1: *Cashflows and symbol for an increasing n-year annuity immediate.*

Time	0	1	2	3…	N
Cashflows		n	n-1	n-2..	1
PV	$(Da)_{\overline{n}\|i}$	nv^n	$(n\text{-}1)v^{n\text{-}1}$	$(n\text{-}2)v^{n\text{-}2}$…	v

Table 5.2: *Cashflows and symbol for a decreasing n-year annuity immediate.*

5.2 Formulae for Present Value: Using Table 5.2 we can verbally derive the formula for the present value of the decreasing annuity (Table 5.4). Imagine a depositor depositing n in the bank at time t=0 and withdrawing 1 at the end of each year. The present value of i) (Row (1)) the deposit of n, minus, the ii) (Row (2)) deposit of 1 at the end of each year, must equal (Row (4)) the interest the bank gives. Hence, we elegantly derive the following equation.

(5.3) $i(Da)_{\overline{n}|i} = nv^n + (n-1)v^{n-1} + \ldots + v = \text{PV Row(1)} - \text{PV Row(2)} = n - a_{\overline{n}|i} \longrightarrow (Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i}$

Time	PV	0	1	2	3	…	n
(1) Cashflows of depositor	n	0	0	0	0	…	0
(2) Withdrawals of depositor	$a_{\overline{n}\|i}$	0	1	1	1	…	1

Total left in Bank		N	n-1	n-2	n-3	…	1
(4) Cashflows of Bank	$i(Da)_{\overline{n}\|i}$		ni	$(n$-1$)i$	$(n$-2$)i$	…	i

Table 5.4: *Verbal argument for derivation for formula for decreasing annuity.*

We can then verbally derive the equation for the present value of an increasing annuity using the cashflows presented in Table 5.6, which gives rise to the following equation.

(5.5) $$(n+1)a_{\overline{n}|i} = (Da)_{\overline{n}|i}+(Ia)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i}+(Ia)_{\overline{n}|i} \longrightarrow (Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}.$$

Time	0	1	2	3…	n
(1) Level		n+1	n+1	n+1…	n+1
(2) Decreasing		n	n-1	n-2…	1
(3) Increasing		1	2	3…	n

Table 5.6*: PV Row (1) = PV Row (2) + PV Row(3) corresponding to the first equality in (5.5).*

Tables 5.4 and 5.6 are useful computationally. For example, Table 5.6 shows how to replace increasing and decreasing annuities with each other in any EOV.

5.3 Calculator TimeValue Line: There is no special method for calculating a decreasing annuity. However, Brovender introduced a computational trick to calculate increasing annuities. This trick is presented in Table 5.7.

N	I	PV	PMT	FV	Calculated BGN mode
5	2	CPT= {0.2791}	-1	5	$\ddot{a}_{\overline{5}|2\%} - 5v^5$
		Divide .02= {13.9537 }			$\frac{\ddot{a}_{\overline{5}|1\%} - 5v^5}{0.02} = (Ia)_{\overline{5}|2\%}$
		x 7 = {97.6761}		0	$7(Ia)_{\overline{5}|2\%}$

Table 5.7: *Illustrative calculation of the Brovender approach to calculating increasing annuities.*

5.4 Six Level Increasing / Decreasing Annuity Functions: We can generate six increasing annuity functions from the basic increasing annuity shown above. These six functions are presented in Table 5.8 along with symbols and formulae.

Time	10	11	...	10+n	11+n	

Cashflow		1	…	n		
Increasing Annuity immediate	$(Ia)_{\overline{n}\|i} = \dfrac{\ddot{a}_{\overline{n}\|i} - nv^n}{i}$					
Annuity due		$(I\ddot{a})_{\overline{n}\|i} = \dfrac{\ddot{a}_{\overline{n}\|i} - nv^n}{d}$				
				$(Is)_{\overline{n}\|i} = (1+i)^n (Ia)_{\overline{n}\|i}$		
					$(I\ddot{s})_{\overline{n}\|i} = (1+i)^n (I\ddot{a})_{\overline{n}\|d}$	
Timeline	10	11…	...	10+n	11+n	…
Cashflow		1…	…	10	11	…
Increasing Perpetuity Immediate	$(Ia)_{\overline{\infty}\|i} = \dfrac{1}{id} = \dfrac{1+i}{i^2}$					
Increasing Perpetuity Due		$(I\ddot{a})_{\overline{\infty}\|i} = \dfrac{1}{d^2} = \left(\dfrac{1+i}{i}\right)^2$				

Table 5.8: *Six increasing annuity functions. The first four are each associated with the same set of cashflows but evaluated at different points of time. The last two are perpetuities*.

Similarly, the decreasing annuity function presented above gives rise to four decreasing annuity functions associated with the same set of cashflows but evaluated at different points of time. These four functions are presented in Table 5.9 with their associated symbols, timelines and formulae.

Time	10	11	12	...	10+n	11+n
Cashflow		*N*	*n-1*	…	1	
Decreasing Annuity immediate	$(Da)_{\overline{n}\|i} = \dfrac{n - a_{\overline{n}\|i}}{i}$					

Decreasing Annuity due		$(D\ddot{a})_{\overline{n}\|i} = \frac{n - a_{\overline{n}\|i}}{d}$				
					$(Ds)_{\overline{n}\|i} = (1+i)^n (Da)_{\overline{n}\|i}$	
						$(D\ddot{s})_{\overline{n}\|i} = (1+i)^n (D\ddot{a})_{\overline{n}\|d}$

Table 5.9: *Four decreasing annuity functions and their associated symbols, timeline cashflows and formulae.*

5.5 Timeline Decompositions: The methods of timeline decomposition have already been presented. However, increasing and decreasing annuities present a special type of annuity that naturally is solved using a decomposition into two timelines. To illustrate we use **QIT#143**.

QIT#143: *Calculate the present value of a perpetuity immediate with successive annual payments of 6, 8, 10, 12, at a 6% annual effective rate.*

The timelines to solve this problem are presented in Table 5.11 immediately below. But how is this Table understood. It is understood through a 4-step process presented in Figure 5.10

Step Number	Activity at this step	Applied to this example: QIT#143
0 (TL_1)	Write down cashflows: $C_1, C_2, C_3 \ldots \leftarrow TL_1$	F=*First* = C_1 =6, S=Second= C_2 =8, C_3 = 10…
1	Compute common difference: $d = S - F = C_2 - C_1 = C_3 - C_2$	8-6=2, 10-8=2, 12-10=2… So $d = 2 = S - F$
2 (TL_2)	Write cashflows for $d(Ia)_{oo}$ This is TL_2	$d(Ia)_{oo} = 2(Ia)_{oo}$ 2, 4, 6, 8, …
3 (TL_3)	Write cashflows for TL_3 $TL_3 = TL_1 - TL_2$	$C_1 = F - d = F - (S - F) = (2F - S) = 2*6-8=4$, C_2 =8-4=4, C_3 = 10-6=4…
4	Write down EOV for TL_1 $PV_3 = PV_1 - PV_2 \rightarrow$ $PV_1 = PV_2 + PV_3$	$PV_1 = d\,(Ia)_{oo} + (F - d)\, a_{oo}$

Figure 5.10: Method and steps for constructing Table 5.11

Time	0	1	2	3	4…
Timeline 1		6 = *F*	8	10	12…
Timeline 2		2 = *d*	4	6	8…
Timeline 3		4 = (*F* - *d*)	4	4	4…

Table 5.11: *Four timelines needed to solve* ***QIT#143.***

Using high-level concepts, we can explain the timelines as follows. We first notice that TL_1 seems to be an increasing annuity, but it does not have the *right* form: 2,4, 6,….

This is a typical issue that arises with increasing annuities. To solve it we follow the 4 steps in Figure 5.10.

Step 1: Take the *common difference* in TL_1; We have as follows:

8-6=2 10-8=2 12-10=2

So, the common difference is 2.

Step 2: We now write TL_2 as an increasing annuity: 2 x1, 2 x 2, 2 x 3, … which does have the proper form of an increasing annuity.

Step 3: Now what? We can evaluate TL_2 but it does not equal TL_1. Therefore, we create *timeline 3:*

$$TL_3 = TL_1 - TL_2$$

So indeed, 6-2=4, 8-4=4, 10-6=4, … and we have a level annuity.

Step 4: To summarize we have $TL_1 = TL_2 + TL_3$ and hence

$$PV_1 = PV_2 + PV_3 = 2(Ia)_{\infty|6\%} + 4a_{\infty|6\%} = \frac{2}{6\% \times \frac{6\%}{1.06}} + \frac{4}{6\%} = 655.56$$

Note: The SOA solution (SIT#143) just contains the last 3 equalities without any timelines. The timelines tell you what to do! Also, note that many textbooks have a special formula for these types of increasing annuities. But their formula is not an internationally accepted actuarial notation. Furthermore, one must calculate the common difference anyway to use their formula. So, it is best to use the above method. The above method also prepares one for new types of problems.

5.6 Summary and Further Examples: Figure 5.11 summarizes the criteria and equation 5.12 summarizes the equation. This equation works on any arithmetically increasing or decreasing sequence whether it is finite or infinite and is very useful

Suppose a sequence $F=C_1$, $S=C_2$, C_3, … satisfies the following

i) There is a *common difference* $d = S\text{-}F = C_2\text{-}C_1 = C_3\text{-}C_2$
ii) The payments are immediate
iii) There are n payments where n may be finite or infinite

Then the present value of these payments one period before the first payment is given by the formula in (5.13). This formula is called the FIND formula since it uses the parameters, *F, d, n, and i.*

Figure 5.12: Criteria for applying the arithmetic formula in (5.13)

(5.13) FIND Formula $PV = (F-d)a_{n|i}+d(Ia)_{n|i}$

If the payments are due then the two annuity symbols should be made due and the formula still holds. The following examples are illustrative

Example 5.14: Find the present value of payments of 5,7,9,11 at t=1,2,3,4. We suppose i = 7%.

Solution: First payment, F=5; second payment, S= 7; $d = S - F$= 7-5=2; Number of payments n = 4. The PV is therefore $(5-2)a_{4|7\%} + 2(Ia)_{4|7\%} = 3 * 3.3872 + 2 * 8.1819 = 26.5255$

Example 5.15: Find the present value of payments of 11,9,7,5 at t = 1,2,3,4. We suppose i = 7%.

Solution: The first payment, F=11; second payment, S= 9; $d = S - F$= 9-11=-2; Number of payments n = 4. The PV is therefore
$(11-(-2))a_{4|7\%} - 2(Ia)_{4|7\%} = 13 * 3.3872 - 2 * 8.1819 = 27.6699.$

5.7 SOA QIT Problems. Problems on increasing and decreasing annuities are presented below. Each problem should be worked through until mastery using the concepts presented above.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

#6 – Increasing Annuity Perpetuity Algebra
#18 – Increasing Annuity Conversion Brovender
#86 – Increasing Level Annuity
#101 – Increasing, Level Annuity, Deferral
#136 – Increasing Annuity, Algebra
#141 – Increasing Annuity, Brovender 🡐 *Poor question – no subproblems*
#142 – Increasing Annuity, Nonstarter, Point Deposit
#143 – Increasing Annuity, Non-starter

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

N05#23 Decreasing - Calculator Too easy

N00#44 Decreasing - (Increasing-Level-Ramp)

N00#20 Level - Increasing

N05#12 Level-Increasing-Perpetuity

M05#14 Level-Decreasing-Ramp

CHAPTER 6
INFLATION, GEOMETRICALLY INCREASING ANNUITIES

6.1 Definition of Inflation: We consider a very simple scenario. Suppose your favorite beverage – beer, orange juice, milk – is selling for 1 today at time t=0. During the next year, *prices may go up.* Suppose next year the price is 1.50. We say that inflation is 50%. The corresponding inflation factor would be 1.5. This simply means that

$$\frac{\text{Price next year}}{\text{Price this year}} = \frac{1.50}{1.00} = 150\%$$

Now suppose that interest is i=75%. Let us calculate the present value, PV, of a beer, OJ, or milk, one year from now under inflation. Table 6.1 summarizes the present value of a beer, milk or OJ that costs 1.

Time / Interest	t = 0	t = 1	Comments
Time	0	1	
Cost of beer, OJ	1	1	No inflation
i = 75%	57 cents = v =1/1.75	1	It costs 57 cents today to save up to buy a beer, milk or OJ one year from now if there is no inflation
Time	0	1	With inflation
Cost of beer, OJ	1	1.5	50% inflation
i = 75%	86 cents = 1.5v =1.5/1.75	1.5	It costs 86 cents today to save up to buy a beer that costs 1 dollar today if there is 50% inflation

Table 6.1: *Cost of beer/OJ one year from now, with and without inflation, if i =75%.*

6.2 Modified interest rate: Table 6.1 shows how to deal with inflation. We must

- Multiply the numerator of 1 by the inflation factor of 1.5 and also
- Simultaneously divide by the interest factor, 1.75, in the denominator.

However, there is another way to approach this. If I am earning 75 cents in the bank on every dollar I have, i=75%, but the cost of items I buy is increasing 50%, then I am not really earning 75% interest. The 50% inflation reduces my interest rate of 75%.

I will now argue that the appropriate modified interest rate is 16.66%. Suppose i'=16.66%. The corresponding interest factor is 1.1666. The corresponding discount factor is v = 1/1.16666. Notice that

$$1 \times v = 1\frac{1}{1.1666} = 0.86$$

In other words, if I started with 86 cents at time t=0, I would end up with 1 at t=1, with 1 being the cost of beer or OJ at time t=0. We say that 16.66% is the *modified interest rate*. It summarizes the two forces of interest and inflation.

The following formula summarizes how to calculate the modified interest rate.

$$\textbf{(6.2)} \qquad \frac{1}{\text{factor of modified rate}} = v_{i'} = \frac{1}{1+i'} = \frac{\text{inflation factor}}{\text{interest factor}} = \frac{1+g}{1+i}$$

6.3 Using Modified Interest Rate in Annuities: Consider the following two problems.

I. *Price a perpetuity immediate of 1000 if interest rates are 4% effective.*
II. *Price a perpetuity immediate of 1000 with an inflation of 1% per year and an effective rate of 4%.*

Solution to Problem I: We have

$$\text{Price} = \text{Present Value} = 1000a_{\overline{\infty}|4\%} = \frac{1000}{.04} = 25000$$

Solution to Problem II: To fully solve this, we look at the timeline of problem II presented below in the Cashflow row of Table 6.3. Table 6.3 then presents 2 ways to calculate the PV, A and B.

Time	0	1	2	3…	*t* …..
Cashflow – Actual payments	Price	1000	1000 x 1.01=1010	1010 x 1.01= 1000 x 1.01^2 = 1020.10	1000 x 1.01^{t-1}….
$\textbf{(A)}PV = \sum_{t=0}^{t=\infty} \frac{1000}{1.04}\left(\frac{1.01}{1.04}\right)^t$		1000/1.04 =961.54	961.54 x1.01/1.04	961.54 x $1.01^2/1.04^2$….	1000 / 1.04 x $(1.01/1.04)^{t-1}$
$\textbf{(B)}PV = \sum_{t=0}^{t=\infty} \frac{1000}{1.04}\left(\frac{1}{1.0297}\right)^t$		1000/1.04 =961.54	961.54 x1/1.0297	961.54 x $1/1.0297^2$….	1000 / 1.04 x $(1/1.0297)^{t-1}$

Table 6.3: *Two ways to compute a geometrically increasing perpetuity* (*inflating-payment perpetuity*).

Let us now explain methods A and B from Table 6.3. Method A simply computes the present value of each payment and sums it. We get an infinite sum corresponding to a geometric series which can be computed by the geometric series formula.

But if we use Method B we need not compute any geometric series. Let us see why.

- We compute the Present Value at t =1. The Present Value at t=0 can then be obtained by applying the discount factor for 4% interest, $v_{4\%}$.
- In computing the present value at t =2 we note that at each step we have an inflation factor and a discount factor. As we saw in Section 6.2, in equation (6.2), whenever we have inflation and interest we can substitute the modified interest. In this case we easily compute using (6.2), for inflation 1% and interest 4%, that

$$\frac{1}{1.0297} = \frac{1.01}{1.04} \qquad \textbf{(6.3)}$$

- This means we can use the amount at t=1, 1000, and keep it level. But we can no longer use 4% since as we saw above we aren't really earning 4%, rather we are losing 1% to inflation, so that we are using the modified interest rate of 2.97%
- But we can compute an annuity immediate of 1000 using a 2.97% interest rate. We have

$$1000\ddot{a}_{\overline{\infty}|2.97\%} = \frac{1000}{\frac{2.97\%}{1.0297}} = 34666.67 \longrightarrow \text{PV @ } 0 = \frac{1}{1.04}\times 34666.67 = 33333.33$$

We computed the answer without using geometric series.

6.4 Summary of method: When you have an annuity with both inflation and interest, as shown in Table 6.4, you can use the steps in Figure 6.5 to avoid geometric series.

Time	0	1…	m	m+1	m+2..	m+n-1
Payments			P	$P(1+g)$	$P(1+g)^2$…	$P(1+g)^{m+n-1}$
Equivalent payment at i'			P	P	P…	P

Table 6.4: *A series of payments that are inflating (increasing geometrically) starting at time m.*

A. Calculate the modified interest rate, i' as given by equation 6.2
B. Then the current value of the n payments from $t=m$ to $t=m+n$-1, is $P\ddot{a}_{\overline{n}|i'}$
C. The present value at 0 is then obtained by a deferral factor with **Final Answer**: $v^m P\ddot{a}_{\overline{n}|i'}$

Figure 6.5: *Steps to calculate the Present Value of the cashflows in Table 6.4.*

6.5 Examples of the method: We present five examples illustrating this method. To keep matters simple we use purely numerical examples. Some problems may require considerable algebra. In all five problems, we use interest rate 4% and inflation rate 1% with a modified rate of 2.97% as shown in (6.2). In all five illustrative problems, we plug in the formula in step C of Figure 6.5. Comments on the solutions are made after presenting all five problems and solutions.

Illustrative Problem 1: A special annuity provides a payment of 1000 at time t=0. Each successive payment is 1% bigger than the previous payment. There are a total of 10 payments. Compute the price of this annuity.

Solution: $1000\ddot{a}_{\overline{10}|2.97\%} = 8796.90$

Illustrative Problem 2: A special annuity provides a payment of 1000 at time t=10. Each successive payment is 1% bigger than the previous payment. There are a total of 10 payments. Compute the price of this annuity.

Solution: $v_{4\%}^{10} 1000\ddot{a}_{\overline{10}|2.97\%} = 5942.87$

Illustrative Problem 3: A special annuity provides a payment of 1000 at time t=1. Each successive payment is 1% bigger than the previous payment. There are a total of 10 payments. Compute the price of this annuity.

Solution: $v_{4\%} 1000\ddot{a}_{\overline{10}|2.97\%} = 8458.56$

Illustrative Problem 4: A special annuity provides payments of 1000 at times t=1,2,3. Starting with the 4th payment, each payment is 1% bigger than the previous payment. The payments continue forever. Compute the price of this annuity.

Solution: $1000a_{\overline{2}|4\%} + v_{4\%}^{3} 1000\ddot{a}_{\overline{\infty}|2.97\%} = 1886.09 + 0.8890 \times 34666.67 = 32,704.64$

Illustrative Problem 5: A special annuity provides payments of 1000 at times t=1,2,3. Starting with the 4th payment, each payment is 1% bigger than the previous payment. The payments continue until the last payment at time t=6. Compute the accumulated value of this annuity at time t=6.

Solution: $1000s_{\overline{2}|4\%} \times 1.04^4 + 1000\ddot{a}_{\overline{4}|2.97\%} \times 1.04^3 = 2040 \times 1.04^4 + 3830.23 \times 1.04^3 = 6,695$

Comments on the solutions to the five illustrative problems:

- Illustrative problem #1: The first payment is at 0 so we don't need a deferral factor. There are n=10 payments and we use the modified rate i'=2.97%. We simply plug in the formula from row (C) in Figure 6.5
- Illustrative problem #2: This problem starts payments at time 10. So, we need a deferral factor for 10 payments. Hence the factor of v^{10}. Note how 4% is used in the deferral factor since there is no inflation for the first 10 periods.
- Illustrative problem #3: Problem #3 is similar to problem #2 except that the deferral is for one period rather than 10 periods. However, Problem #3 can be formulated using *immediate* vs. *due* terminology. When solving these problems always emphasize where the first payment undergoing inflation occurs.
- Illustrative problem #4: This problem has several interesting features
    - The number of payments geometrically increasing is infinite. That is n=*oo*. We simply substitute infinity into the formula in row (C) in Figure 6.5.
    - The first payment *undergoing inflation* is 1000 at time t=3. Do not confuse this with the first *inflated* payment, 1000 x 1.01, which occurs at time t=4.
    - The first two payments form a separate timeline; they are a level annuity with 2 payments. They are computed at 4% using an immediate annuity.
- Illustrative problem #5: This problem has several interesting features:
    - You need a deferral factor from time t=2 to t=6, 1.04^4
    - You cannot use s-dot for the second annuity. You must use a-dot with a deferral factor of 1.04^3. *Always, first accumulate at the time of the first payment and then defer.*

6.6 Arithmetically Increasing vs. Geometrically Increasing: A problem may speak about payments that are *increasing.* How do you determine if the increase is arithmetic or geometric?

- Arithmetic: If the *amount* is increasing, you use *(Ia)*, an arithmetically increasing annuity. A sample problem may start *An annuity pays X at t and then increases payments by an amount 100 each year*.
- Geometric / Inflation: If the *per-cent* is increasing, you use the inflation methods of this chapter. A sample problem may start *An annuity pays X at t; each succeeding payment is 2% greater than the previous.*

6.7 Special Case when Inflation and Interest are Equal: If $i=g$, then $1+g = 1+i$, implying $1+i' = (1+i)/(1+g)=1$; in other words $i'=0$. In this special case

(6.4) $$a_{n|0\%} = n$$

because the PV of 1 at any time t is simply 1 (there is no discounting).

6.8 Special case for a perpetuity immediate: In the special case of
1) a perpetuity
2) immediate
3) with a payment of 1 at t=1,
4) increasing by g every year

The formulas of this section reduce to a very simple elegant form for the PV

(6.5) $$PV = 1 / (i - g)$$

6.9 SOA QIT Problems: Problems on inflation, or geometrically increasing, annuities are presented below. Each problem should be worked through until mastery using the concepts presented above.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:
#14) Perpetuity – Level – Inflation
#31) Annuity – Inflation
#60) Inflation – Inflation
#102) College Funding – Inflation – Inflation
#103) Perpetuity – Inflation – Conversion
#144) Inflation – Perpetuity
#145) Level-Inflation (Ramp)
#146) Inflation – Deflation – Perpetuity

FROM THE ARCHIVED EXAMS: No solution posted.
SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)
https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx
N01#5 – Level Inflation Perpetuity N05#9 –Level Inflation Perpetuity

CHAPTER 7

Payment Period vs. Interest Period

7.1 Overview: This chapter covers a number of annuities where the payment period and interest period are different. There are special symbols for such annuities. There are also simple methods of converting these annuities to more traditional annuities. Very often these annuities are described with special English phrases with specific meaning. Table 7.1 summarizes these annuities, their symbols, their English descriptions, their timelines, formulas for them, how to approach them by conversion of interest rates.

Table 7.2 takes each symbol in Table 7.1 and gives similar symbols which you must know. Table 7.2 leaves their English description, timeline and method of approach to you (It follows similar principles to what we have done earlier)

7.2 Special English Phrases: Notice that the phrase in row 1 of Table 7.1, *annual payment of 1200 payable 12 times a year* means a payment of 100 a month.

The two bullets below describe two ways of indicating interest. They are both very common.

- Effective annual rate of 4%: Then $i = 4\%; \delta = ln(1.04) = 0.0392$
- Rate of 4% convertible continuously: Then $\delta = 4\%; 1 + i = e^{\delta} = 1.0408$.

7.3 Illustrative Problem: You are given the following information about a bank account with interest rate 3%.

- The account currently has 10,000
- Deposits of 5000 are made (by generous parents) at the beginning of each year for 2 years
- Continuous withdrawals of *Y* per year are made for the four years someone is in college.

Calculate *Y*.

Solution: One can feel overwhelmed by a problem like this. But recall, that whenever you feel overwhelmed simply create enough timelines. In this case, three timelines are created one for each bullet.

Timeline 1	0			
Cashflow	10000			

Timeline 2	0	1	2	3	4
Cashflow	5000	5000			

Timeline 3	0	1	2	3	4
Cashflow	Continuous	Withdrawal	Of *Y* per year		

The present value of these 3 timelines are easy to compute. For timeline 3 we make use of Table 7.1

Timeline 1: $PV_1 = 10000$
Timeline 2: $PV_2 = X\ddot{a}_{\overline{2}|3\%} = 1.9709X = 1.9709 \times 5000 = 9854.37$

Timeline 3: $PV_3 = Y\bar{a}_{\overline{4}|3\%} = Y\dfrac{1 - v_{3\%}^4}{\delta} = 3.7726Y$

In Timeline 3 we have used the fact that $1.03 = 1 + i = e^{\delta} \longrightarrow \delta = \ln(1.03) = 0.0296$

To complete the problem think *inflow* and *outflow*

- Inflow: 10000 and deposits of 5000
- Outflow: Withdrawals of *Y*

So 10000 + 9854.37 = 3.7726*Y* ➔ *Y*=5262.78

7.4 Derivations and Approaches: Although Table 7.1 gives both formulae and approaches, very often the approaches, not the formulas, are quicker and simpler. *In practice, simply use the three/ four principles bulleted below and conversion.* You however are responsible to know the English phrases and symbols introduced.

- An accumulated annuity is simply $(1+i)^n$ times the present value annuity
- A PV of an annuity due is obtained by multiplying the corresponding annuity immediate by a discount factor to an appropriate power.
- A perpetuity is the limit of an *n*-year annuity as *n* goes to infinity
- (Advanced: A continuous annuity is the limit of an *m-th* ly annuity as *m* goes to infinity)

Some illustrative derivations may be helpful.

Example 1: You want the accumulated value of anything. **Solution**: $AV = (1+i)^n PV$

Example 2: Compute the PV of a 5-year annuity paying 2 at the end of each month at 3% effective annually.

Solution: $2a_{\overline{60}|j}$, $(1+j)^{12} = 1.03$

Example 3: Compute the AV of a 5-year annuity paying 2 at the end of each month at 3% effective annually.

Solution: $(1+i)^5 \times 2a_{\overline{60}|j}$, $(1+j)^{12} = 1.03$

Example 4: Compute the PV of a 5-year annuity paying 2 at the beginning of each month at 3% effective annually.

Solution: $(1+j) \times 2a_{\overline{60}|j}$, $(1+j)^{12} = 1.03$

Explanation: The above expression pays 1+j at the *end* of each month. But 1+j at the *end* of each month is actuarially equivalent to 1 at the beginning of each month. For a year use 1+i; for a month use 1+ j. Hence, we may conclude.

Example 5: Give the formula for a continuously increasing increasing annuity.

Solution: Here, it is easier to memorize a formula than derive: $(\bar{I}\bar{a})_{\overline{n}|i} = \frac{\bar{a}_{\overline{n}|i} - nv^n}{\delta}$

Example 6: Give the formula for a continuously increasing continuous perpetuity.

Solution: In the formula for example 5, let n go to infinity. We have

$$\lim_{n\longrightarrow\infty} (\bar{I}\bar{a})_{\overline{n}|i} = \lim_{n\longrightarrow\infty} \frac{\bar{a}_{\overline{n}|i} - nv^n}{\delta} = \lim_{n\longrightarrow\infty} \frac{1}{\delta}\frac{1-v^n}{\delta} = \frac{1}{\delta^2}$$

7.5 Approximations and Intuitions about Continuous Annuities: A good way to *think* of a continuous annuity of say 1 dollar *payable continuously* for a year at force of interest δ is to approximate it by a discrete annuity paying 1/365 of a dollar every day and effective rate $\frac{\delta}{365}$. We can see this vividly if we compare the numerical value of the two payment schemes. Let us suppose that the force of interest is 2%, that the payment period is for 3 years and that the payment is 4.

PV of a continuously payable annuity of 4 for 3 years at interest 2% convertible continuously equals $4\bar{a}_{\overline{3}|\delta=2\%} = 4\frac{1-e^{-3\times 2\%}}{2\%} = 11.65$

PV of $\frac{4}{365}$ a day, at daily compound rate of $\frac{2\%}{365}$ for 3 years equals $\frac{4}{365} a_{\overline{365\times3}|\frac{2\%}{365}} = 11.65$

<table>
<tr><th>Symbol</th><th>English Description</th><th>Timeline</th><th>How to Approach</th></tr>
<tr>
<td>$a_{\overline{n}|i}^{(m)}$</td>
<td>n-year annuity immediate at annual interest rate i, with an annual payment of 1, payable monthly.</td>
<td><table>
<tr><td>Time</td><td>0</td><td>$1/m$</td><td>$2/m$…</td><td>nm/m</td></tr>
<tr><td>Payment</td><td></td><td>$1/m$</td><td>$1/m$…</td><td>$1/m$</td></tr>
</table></td>
<td>$(1+j)^m=1+i$
j, rate per m-th of year. Formula is
$\frac{1}{m} a_{\overline{mn}|j}$
Alternate formula
$\frac{1-v_i^n}{i^{(m)}}$</td>
</tr>
<tr>
<td>$(I^{(m)}a)_{\overline{n}|i}^{(m)}$</td>
<td>n-year increasing annuity immediate with payments increasing m-th ly with annual interest rate i (with 1st payment of $1/m^2$)</td>
<td><table>
<tr><td>Time</td><td>0</td><td>$1/m$</td><td>$2/m$…</td><td>nm/m</td></tr>
<tr><td>Payment</td><td></td><td>$1/m^2$</td><td>$2/m^2$…</td><td>nm/m^2</td></tr>
</table></td>
<td>$(1+j)^m=1+i$
j, rate per m-th of year. Formula is
$\frac{1}{m^2}(Ia)_{\overline{mn}|j}$
Alternate formula
$\frac{\ddot{a}_{\overline{n}|i}^{(m)} - nv_i^n}{i^{(m)}}$</td>
</tr>
<tr>
<td>$(Ia)_{\overline{n}|i}^{(m)}$</td>
<td>n-year annuity immediate with level payments each interest period but increasing annually (with first payment of $1/m$)</td>
<td><table>
<tr><td>Time</td><td>0</td><td>$1/m$…1</td><td>$1+1/m$…2</td><td>$n-1+1/m$…n</td></tr>
<tr><td>Payment</td><td></td><td>$1/m$…</td><td>$2/m$…</td><td>n/m</td></tr>
</table></td>
<td>Special method discussed in Chapter 8</td>
</tr>
</table>

$\bar{a}_{\overline{n}\vert i}$	*n*-year annuity *continuously paying 1 per year* at annual rate *i*	There is no timeline. The idea is that the annuity pays *dt* at teach point of time. Therefore, its present value is $\int_{t=0}^{t=n} v^t dt$ Similarly if a continuously increasing annuity pays *t dt* at time *t*, its present value is $\int_{t=0}^{n} tv^t dt = \int_{t=0}^{n} te^{-\delta t} dt$	$\frac{1-v^n}{\delta} = \frac{1-e^{-\delta n}}{\delta}$ Recall $e^{\delta} = 1 + i$

Table 7.1**:** *Several annuities with different payment and interest periods.*

Symbol	English Description
$a_{\overline{n}\|i}^{(m)}$	$\ddot{a}_{\overline{n}\|i}^{(m)}$, $a_{\overline{\infty}\|i}^{(m)}$, $\ddot{a}_{\overline{\infty}\|i}^{(m)}$, $\ddot{s}_{\overline{n}\|i}^{(m)}$, $s_{\overline{n}\|i}^{(m)}$,
$(I^{(m)}a)_{\overline{n}\|i}^{(m)}$	$(I^{(m)}\ddot{a})_{\overline{n}\|i}^{(m)}$, $(I^{(m)}a)_{\overline{\infty}\|i}^{(m)}$, $(I^{(m)}\ddot{a})_{\overline{\infty}\|i}^{(m)}$, $(I^{(m)}s)_{\overline{n}\|i}^{(m)}$, $(I^{(m)}\ddot{s})_{\overline{n}\|i}^{(m)}$,
$(Ia)_{\overline{n}\|i}^{(m)}$	$(I\ddot{a})_{\overline{n}\|i}^{(m)}$, $(Ia)_{\overline{\infty}\|i}^{(m)}$, $(I\ddot{a})_{\overline{\infty}\|i}^{(m)}$, $(Is)_{\overline{n}\|i}^{(m)}$, $(I\ddot{s})_{\overline{n}\|i}^{(m)}$,
$\bar{a}_{\overline{n}\|i}$	$\bar{a}_{\overline{\infty}\|i}$, $\bar{s}_{\overline{n}\|i}$, $(\bar{I}\bar{a})_{\overline{n}\|i}$, $(\bar{I}\bar{s})_{\overline{n}\|i}$, $(\bar{I}\bar{a})_{\overline{\infty}\|i}$,

Table 7.2: *For each symbol in Table 7.1, symbols for several similar annuities are listed.*

7.6 SOA QIT Problems: Problems on increasing and decreasing annuities are presented below. Each problem should be worked through until mastery using the concepts presented above.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

#138) Continuous - Continuos

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

CHAPTER 8

Nested Annuities

8.1 Overview: This chapter covers annuities, periodic patterns, where there are patterns within patterns. Some examples are as follows:

- 1,1,2,2,3,3,…
- 1,2,3,1,2,3,1,2,3,1,2,3,…
- 1,1,2,1,1,2,1,1,2,…
- 1,1,1,1.03,1.03,1.03, 1.03^2,1.03^2,1.03^2,…
- 1,2,3,11,12,13,21,22,23,31,32,33,…

There are many similar patterns. In each pattern, one can see a small repeating pattern nested inside a bigger repeating pattern.

These annuities are difficult because they must be done by *method* not by *formula.*

Throughout this section i=5%. Various conversions are as follows:

$$i = 5\%, \qquad (1+i)^2 = 1+j = 1.1025 \qquad (1+i)^3 = 1+k = 1.1576.$$

8.2 The Nested Annuity Method: The method to be used in all problems is summarized in Figure 8.1.

A. Create a timeline for the entire payment scheme
B. Identify the smaller pattern, typically this is the pattern in the first few consecutive payments
C. Now create a 2^{nd} timeline consisting of repeated occurrences of the smaller pattern identified
D. Now identify the PV of the 2nd timeline. Be careful to change interest rates if necessary.
   This is your answer
   CAUTION: ***Notice how both i and j are used in answer; both due and immediate. Always check***

Figure 8.1: *General method for nested annuities.*

We will now apply *the method* to several of the examples in 8.1.

8.3 The Annuity 1,1,2,2,3,3,4,4,…

The sequence of timelines and present values are presented in Table 8.2. The letter numberings correspond to the letter numberings in Figure 8.1. This will enable the student to see *how* Figure 8.1 *applies to* this problem.

Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**…**
Original Timeline (**A**)		1	1	2	2	3	3	…				
Smaller Pattern (SP) **(B)**		SP1	SP1	SP2	SP2	SP3	SP3	…				
(C) = PV(B) – TL2	$a_{\overline{2}\rvert}$		$2a_2$		$3a_{\overline{2}\rvert}$			…				
(D) $(I\ddot{a})_{\overline{\infty}\rvert j}a_{\overline{2}\rvert i}$												

Table 8.2: *Application of* the method *of Figure 8.1 to the payment pattern, 1,1,2,2,3,3,….*

We can briefly summarize this as follows: First, after writing down the entire timeline (A), we noticed a *smaller* pattern of a 2-payment annuity (B) every two years. This allows us (C) to rewrite the original timeline as an increasing series of payments every 2 years beginning at 0. The 2-payment annuities reflect the two payments on timeline B. We can factor out the two-payment annuity and are left with an increasing annuity due (D). We can easily calculate the PV of this annuity (E). The PV of the original timeline is the product of the 2-payment annuity and the increasing annuity.

To emphasize technique, throughout this chapter, numerical values will not be calculated.

8.4 Table 8.2 Revisited Step by Step: To help students, I will repeat the derivation and go through Table 8.2 step by step. Imitating this technique will help you when doing problems.

STEP A:

Simply write down the pattern completely *without any patterns.* You obtain

Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**…**
Original Timeline (**A**)		1	1	2	2	3	3	4	4	5	5	…

Table 8.2A: *Step A of Table 8.2*

STEP B:

Try and identify a small pattern *without also identifying the bigger pattern.* You obtain:

Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**…**
Original Timeline (**A**)		1	1	2	2	3	3	4	4	5	5	
Smaller Pattern (SP) **(B)**		SP1	SP1	SP2	SP2	SP3	SP3	SP4	SP4	SP5	SP5	

Table 8.2B: *Step B of Table 8.2*

STEP C

This is the part where some of you get stuck in the homework. You are trying to do too much. You are trying to do the bigger pattern. It is not yet time. You found the smaller pattern! So place each pattern in its appropriate place. We obtain

Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**…**
Original Timeline (**A**)		1	1	2	2	3	3	4	4	5	5	
Smaller Pattern (SP) **(B)**		SP1	SP1	SP2	SP2	SP3	SP3	SP4	SP4	SP5	SP5	
(C) = PV (B)	$a_{\overline{2}\rceil i}$		$2a_{\overline{2}\rceil i}$		$3a_{\overline{2}\rceil i}$		$4a_{\overline{2}\rceil i}$	…	$5a_{\overline{2}\rceil i}$	…		

Table 8.2C: *Step C of Table 8.2*

Note: You cannot add up $1a_{\overline{2}\rceil i} + 2a_{\overline{2}\rceil i} + 3a_{\overline{2}\rceil i} + 4a_{\overline{2}\rceil i} \ldots$ Why can't you add them. Because they take place at different times. *It is therefore important to place each annuity in its proper place.* Here for example, each 2-payment annuity is the present value of two payments and this present value occurs one period prior to the first payment. Look over Table8.1C and make sure you understand this.

STEP D:

You can see that the timeline in C is an increasing annuity. This allows solving

Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**…**
Original Timeline (**A**)		1	1	2	2	3	3	4	4	5	5	
(C)	$a_{\overline{2}\|i}$		$2a_{\overline{2}\|i}$		$3a_{\overline{2}\|i}$		$4a_{\overline{2}\|i}$	…	$5a_{\overline{2}\|i}$	…		
(D) $(I\ddot{a})_{\overline{\infty}\|j}a_{\overline{2}\|i}$												

Table 8.2D: *Step D of Table 8.2*

8.5 An alternate method for finding PV of 1,1,2,2,3,3,…. The alternate method is summarized in Table 8.3. The alternate method is also based on breaking up timelines into multiple timelines (but not nested timelines)

Time0	-1	0	1	2	3	4	5	6	7…
TL0			1	1	2	2	3	3	…
TimeLine1		0		1		2		3	…
PV1 $(Ia)_{\overline{\infty}\|j}$				1		2		3	…
Timline2	0		1		2		3		…
PV2 $(Ia)_{\overline{\infty}\|j}$			1		2		3		…

Table 8.3: *Alternate method for annuity 1,1,2,2,3,3,…*

Note that timeline 2 is computed at -1 rather than 0 because an immediate PV is calculated one period prior to the first payment. We therefore have the following EOV computed at time t=0.

EOV: $PV = PV_1 + PV_2 = (Ia)_{\overline{\infty}|j} + (1+i)(Ia)_{\overline{\infty}|j}$

Note the deferral factor of (1+i) to push *from t* =-1 *to t* = 0.

8.6 Why *method* vs. *formula*: Why have we called the above computations *method* vs. *formula?* The reason is because there is no one formula. *The method* relies on a combination of two or more annuities.

- But those annuities could be anything: level, increasing, inflation, etc.
- Furthermore, each or either of these annuities could be either immediate or due.
- The annuities can be combined by products, sums or even quotients.
- There might be double or triple nesting
- One or both annuities might be finite or infinite.

It is therefore important when using *the method* to carefully decide what each annuity is – level, increasing, inflation – whether it is immediate or due – whether the two annuities are combined by products, sums or quotients – whether there is double or triple nesting – whether the annuities are finite or infinite. The next few examples illustrate some of these variations.

8.7 The annuity 1,1,1, 1.03, 1.03,1.03, 1.03^2, 1.03^2, 1.03^2, …. We suffice with the table of 5 steps and a final EOV. The 5 steps in Table 8.4 correspond to Figure 8.1. A narrative is provided afterwards.

Before stating the final answer, we summarize: We start (Step A) with the payment pattern 1,1,1,1.03,1.03,1.03, 1.03^2,1.03^2, 1.03^2. We notice a level pattern of 3, the smaller pattern (B). This gives rise to a factor of a 3-payment level annuity (C). We are then left with an inflation annuity but at 3-year intervals. We must treat this through inflation methods. Notice the subtlety: The inflation, 3%, is once every *three* years; so we divide not by the annual rate, 5%, but by the 3-year rate 15.76%.

PV of original annuity $= a_{\overline{3}|i} \times \ddot{a}_{\overline{\infty}|i'}$, $\quad i = 5\%; \quad (1+i)^3 = 1+k; \quad \frac{1}{1+i'} = \frac{1.03}{1+k}$

Time	0	1	2	3	4	5	6	7	8	9…
Time 0 Original Timeline (A)		1	1	1	1.03	1.03	1.03	1.03^2	1.03^2	1.03^2 …
Smaller Pattern (B)		SP1	SP1	SP1	SP2	SP2	SP2	SP3	SP3	SP3
C = PV(B)	$a_{\overline{3}|i}$			$1.03a_{\overline{3}|i}$			$1.03^2a_{\overline{3}|i}$			…
(D) Evaluate PV of previous timeline using geometric method with $\frac{1}{1+i'} = \frac{1.03}{1.1576}$	$\ddot{a}_{\overline{\infty}|i'}a_{\overline{3}|i}$									

Table 8.4: *Application of the method*, Figure 8.1, *to 1,1,1,1.03,1.03,1.03, 1.03^2,1.03^2, 1.03^2,…*

8.8 Three methods to approach the annuity 1,1,2,1,1,2,1,1,2… We have presented *the method* to deal with nested annuities. However, there are a variety of other methods available to you. We illustrate by showing three methods to evaluate 1,1,2,1,1,2,1,1,2…

Method A: *The Method:* We notice the smaller pattern, 1,1,2 (Step B). We can evaluate this as $PV_1 = a_{\overline{2}|i} + 2v_i^3$. Notice how we had to use a point evaluation and a 2-payment annuity. We can create an annuity with this PV_1 at points 0,3,6,9,12,… (Step C). At step D we notice the level annuity of three year payments and obtain.

PV of 1,1,2,1,1,2,... $= \left(a_{\overline{2}|i} + 2v_i^3\right) \times \ddot{a}_{\overline{\infty}|k}$

By the way, you could also use $v_i + v_i^2 + 2v_i^3$ for the parenthetical expression.

Method B: *Deferral Factors:* As in section 8.4, we can use deferral factors paying special attention to i) interest rate per period, ii) where PV are evaluated. We simply break the annuity up into three annuities 1,1,1,1,… and 1,1,1,1,… and 2,2,2,2…. As in Section 8.4 we obtain

PV of 1,1,2,1,1,2,1,1,2,... $= 2a_{\overline{\infty}|k} + (1+i)a_{\overline{\infty}|k} + (1+i)^2 a_{\overline{\infty}|k}$

Method C: *Subtraction vs. deferral method:* This is summarized in Table 8.5. Table 8.5 presents a general approach to avoid deferrals by using addition and subtraction of timelines.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12…
TLine0		1	1	2	1	1	2	1	1	2	1	1	2…
TLine1		1	1	1	1	1	1	1	1	1	1	1	1…
TLine2 = TL0 – TL1				1			1			1			1…

Table 8.5: *Alternate approach based on subtraction of timelines vs. deferrals.*

$PV_0 = PV_1 + PV_2 = a_{\overline{\infty}|i} + a_{\overline{\infty}|k}$

8.10 SOA QIT Problems: Problems on nested annuities are presented below. Each problem should be worked through until mastery using the concepts presented above.

Problem 93 in the QIT is typical *level-monthly-increasing-yearly* problem. Here is the problem. Below it is the solution

Seth has two retirement benefit options.

His first option is to receive a lump sum of 374,500 at retirement.

His second option is to receive monthly payments for 25 years starting one month after retirement. For the first year, the amount of each monthly payment is 2000. For each subsequent year, the monthly payments are 2% more than the monthly payments from the previous year.

Using an annual nominal interest rate of 6%, compounded monthly, the present value of the second option is P.

Determine which of the following is true.
(A) P is 323,440 more than the lump sum option amount.
(B) P is 107,170 more than the lump sum option amount.
(C) The lump sum option amount is equal to P.
(D) The lump sum option amount is 60 more than P.
(E) The lump sum option amount is 64,090 more than P.

SOLUTION: We use the steps in Figure 8.1. Notice that the lump sum is received at the beginning of his retirement just as the series of payments is present valued at the beginning of his retirement. So we must compute the present value of the nested annuity and compare it with the lump sum.

STEP A: We have payments of 2000 at t=1,2,3…12. Then payments of 2000(1.02) at t=13,14,15,…,24. Etc.

STEP B: We recognize the 12 payments of 2000 as a level annuity. Similarly the 12 payments of 1.02 x 2000 is a level annuity. And similarly, the 12 payments of 1.02^2 x 2000 is a level annuity.

STEP C:

- The 12 payments of 2000 have a PV at 0 of $2000a_{\overline{12}|0.5\%}$ Similarly
- The 12 payments made at t= 13 through 24 of 2000(1.02) have a present value at time t=12 of $2000(1.02)a_{\overline{12}|0.5\%}$
- The 12 payments made at t=*25-36* of $2000(1.02)^2$ have a PV at time t=24 of $(1.02)^3 2000a_{\overline{12}|0.5\%}$

STEP D: So the final answer using inflation methods is $\ddot{a}_{\overline{25}|i'} 2000a_{\overline{12}|0.5\%}$

We still have to do the computations. First we must figure out what i' is. Notice that the inflation is 2% *per year* but the interest is 0.5% *per month.* So the interest factor per year is $1.005^{12} = 1+j$ =1.0617 and the modified interest rate is given by 1/(1+i') = 1.02/1.0617 = 1/1.0419 so i' = 4.0861%

We can finish the computations using the TV line

```
N-----I-------PV-------------PMT----------FV
12    0.5     CPT =23,238    -2000        0
25    4.0861  CPT=374,444    23,238 (From PV of previous line)
```

As can be seen, the PV, 374,444 is less than 60 from the lump sum of 374,500.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

#93) Inflation- Level Month – Increasing Yearly

#139) Nested Annuities

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

CHAPTER 9

Outstanding Loan Balance & Balloon Payments

9.1 Overview: In this chapter we deal with loans. Throughout the chapter we deal with a loan of 1000 which is paid back with five equal payments at the end of the year. The effective interest rate is 10%. You can all do this: The 1000 is an *inflow*, something you receive. The five payments are *outflows*, something you pay. The inflow must equal the outflow. We have the following timeline and EOV.

Time	0	1	2	3	4	5
Cashflow	-1000	R	R	R	R	R

Table 9.1: *Standard timeline for a loan with payment by level annual end of year payments. i =10%.*

EOV: $1000 = Ra_{\overline{5}|10\%}$

N	I	PV	PMT	FV	No BGN
5	10	-1000	CPT=263.80=R	0	

Table 9.2: *Standard TV line for calculating the level end-of-year payments for a loan.*

9.2 What is New?: You can already do this. The following two questions are typical of what is new.

Question A: *At time t=2 right after the payment of R, I decide to make an extra payment of 156.03. I then want to refinance the loan to last only 2 more periods. The bank charges me now 12%, because I am changing the terms of the loan. What is my new level end-of-year payment.*

Question B: *Although I can make five end of year payments of 263.80. I offer to make 4 payments of 300. I can then finish paying off the loan in 3 ways.*

(B1) I can make a smaller payment at time t=5. What is that smaller payment?

(B2) I can make an extra payment at time t=4. What is that extra payment?

(B3) Instead of paying 300 at time t=4, I can make a bigger payment at time t=4. What is that payment?

The following 3rd question is a variant of Question B.

Question C: *I decide to pay off a loan of 1000 with a collection of level end-of-year payments of 300 and a final smaller payment* (Notice how you are not told *how many* payments of 300 are made). *Calculate the final payment.*

This chapter will show you how to answer these questions. What is the hard part? The hard part is that my 0 point keeps on changing. For example, if I pay 156.03 at time t=2 then I have made t=2 my new 0. So, the hard part of all these loan questions is the constant shifting of 0 points.

On a final note: Extra payments at the end of a loan series that differ from the level payments are called *balloon* payments. We will not use this terminology but you should know it.

9.3 Four Methods of Computing OLB, Outstanding Loan Balance: A crucial component in any question about loans is computing the outstanding loan balance at a given point of time. Intuitively, the OLB_t is how much you would have to pay at t to completely pay off the loan. OLB is a very important concept. Before answering questions *A,B,* and *C* we show four methods of computing OLB.

Prospective Method: At time t=2, I have left to make 3 payments of R starting one year from t=2. That is simply a new timeline. Let us think about it by expanding Tables 9.1 and 9.2. The new EOV is as follows.

EOV: $OLB_2 = Ra_{\overline{3|}10\%}$

We can modify tables 9.1 and 9.2 to tables 9.1B and 9.2B to deal with the EOV for the OLB.

Time	0	1	2	3	4	5
Cashflow	-1000	R	R	R	R	R
New Timeline						
Time			0	1	2	3
Cashflows			$-OLB_2$	R	R	R

Table 9.1B: *Timeline for OLB at time t=2.*

The EOV which we can read from Table 9.1B is as follows:

$OLB_2 = Ra_{\overline{3|}i}$

N	I	PV	PMT	FV	No BGN
5	10	-1000	CPT=263.80=R	0	
3	Keep	CPT=656.02	Keep 263.80	0	

Table 9.2B: *I make one change: The 5 becomes a 3 (3 more payments left) and I can compute OLB_2*

The modified EOV reflects an equation at t=2 between outflows of payments and the outstanding loan balance which is considered an inflow. You can also think of this as a new loan.

EOV: $OLB_t = Ra_{\overline{n-t}|} \longrightarrow \quad 656.02 = 263.80a_{\overline{3}|i}$

In applying this equation t is the *time* of the payment not the *number* of the payment. So in a due situation the (t+1)*st* payment occurs at *time t.*

This method of computing OLB is called the *prospective* method since we look forward to compute the OLB.

Retrospective method: But we can also look back. The 1000 loan grows in two years to $1000(1.1)^2$. We subtract from this the accumulated value annuity of two payments of 263.80. Here is the formula

$$L(1+i)^t - Rs_{\overline{t}|i} \longrightarrow 1000 \times 1.1^2 - 263.80s_{\overline{2}|10\%} = 656.03$$

Notice how we continuously think of where we are. The 1000 is no longer at t =0 but is at t=2.

Sales Approach: The OLB_2 represents how much someone would *pay* to buy the loan repayments. In exchange for spending OLB_2 the person would receive payments and then might resell it. Suppose the person resells at t=4. To understand the timeline and EOV recall that OLB_2 represents *sale* price, what someone is spending (outflow); while the payments of R are inflow, something received; the payment at t=4 is something received. We have

$$OLB_2 = Ra_{\overline{2}|10\%} + OLB_4 v^2_{10\%} \longrightarrow 656.03 = 263.80a_{\overline{2}|10\%} + OLB_4 v^2_{10\%} \longrightarrow OLB_4 = 239.82$$

The timeline is presented in Table 9.3

N	I	PV	PMT	FV
2	10	-656.03	263.80	CPT=239.82

Table 9.3: *TV line for computing OLB_4 using the sales approach.*

Cashflow approach: We can also use a step by step cashflow analysis. This is presented in Table 9.4.

Time	0	1	2
Cashflow	-1000	1000 x 1.1 = 1100	836.20 x1.1 = 919.82
		-263.80	-263.80
		Net: 836.20	NET: 656.03= OLB_2

Table 9.4: *Cashflow approach to calculating OLB_2.*

9.4 Using the Four Formulae: We have presented in Section 9.3 four formulae. Each formula is useful in a different situation.

- Use the prospective formula if you know *n* and *t*
- Use the retrospective formula if you know *t* but don't know *n*
- Use the sales formula if you neither know *t* nor *n*
- Use the cashflow approach if you only have a few rows to deal with.

9.5 Solution to Question A: We can now answer Question *A* from section 9.2. Simply calculate OLB_2. Then subtract off the extra payment of 156.03. We have left 500. This 500 is the new OLB_2. There are now 2 payments left at 12%. This is simply a new refinancing problem. We modify Table 9.1B to Table 9.1C and similarly modify Table 9.2B to Table 9.2C.

Time	0	1	2	3	4	5
Cashflow	-1000	R	R	R	R	R
New Timeline						
Time			0	1	2	3
Cashflows			$-OLB_2$=656.03	R	R	R
Cashflow new At 12%			OLB_2-156.03=500	R	R	

Table 9.1C: *Timeline for OLB at time t=2.*

Of course, the EOV is simply the basic formula.

$$OLB_2^{(New)} = OLB_2 - 156.03 = Ra_{\overline{2}|12\%}$$

N	I	PV	PMT	FV	No BGN
5	10	-1000	CPT=263.80=R	0	
3	Keep	CPT=656.03	Keep 263.80	0	
		-156.03			
		500 left at *t*=2			
2 payments	12	-500	CPT=295.85	Keep 0	

Table 9.2C: *Calculating the refinanced loan*

The refinancing has an EOV corresponding to a 0-point at the time of refinancing. In this case *t*=2 becomes the new 0 point as shown in Table 9.1C. The EOVs for *each* refinancing typically come in pairs, one for the OLB and one for the computation of the new payment. In the example we are studying, we obtain the following two equations.

$$OLB_2 = Ra_{\overline{3}|i} \longrightarrow 656.03 = 263.80a_{\overline{3}|i}; \qquad 500 = (OLB_2 - 156.03) = R'a_{\overline{2}|12\%} \longrightarrow R' = 295.85$$

The process of paying off part of a loan and possibly changing terms is called refinancing. A refinance problem can contain any, some, or all of the following:

- Special one-time payment
- Number of remaining periods changes
- Effective rate changes
- Level Payment changes.

9.6 Calculating Final Payments: Let us deal with questions B1-B3. Suppose I make 4 payments of 300 on a 1000 loan. What do I have left. The key here is to calculate the OLB_4. The retrospective method looks at the fact that the 1000 loan has grown in 4 years but the accumulated value of four payments of 300 must be subtracted.

Retrospective Method: $1000 \times (1.1)^4 - 300 s_{\overline{4}|10\%} = 71.80$

Table 9.5 exhibits calculation of this on the TV line. Notice the signs: 1000 is an inflow while the 300 and 71.80 are outflows.

N	I	PV	PMT	FV
4	10	-1000	300	CPT=71.80

Table 9.5: *TV calculation of OLB_4 from four end-of-year payments of 300 on a loan of 1000 at 10%*.

At this point we do not use formulas but heuristic reasoning. Here are the arguments.

- Answer to question (B2) in Section 9.2: At time *t*=4, after payment of the 300 payment I have an OLB of 71.80. So, the *extra smaller payment to be made besides the regular 300 payment is 71.80.* Simple enough.
- Answer to question (B3) in Section 9.2: Alternatively, to pay off my loan at *t*=4 I must pay one bigger payment of 300+71.80 = 371.80
- Answer to question (B1) in Section 9.2: Suppose I decide to wait till *t*=5 to pay off the loan. Well, at *t*=4 I owe 71.80. In one year (from *t*=4 to *t*=5) that debt of 71.80 grows to 71.80 x 1.1 = 78.98. So, the smaller payment to pay off the loan at *t*=5 is 78.98

The calculations are all straightforward. What is new is the constant shifting of the 0 point. Where am I starting, where am I ending, and how will the money grow?

9.7 Solution to Question C: But suppose we are not told how many periods we are paying 300? How do we approach the problem? This can elegantly be done on the TV calculator as shown in Table 9.6

N	I	PV	PMT	FV
CPT=4.2542	10	-1000	300	0

Table 9.6A: *The table shows that it will take 4.2542 years to pay off a loan of 1000 using level payments of 300*

Table 9.6A corresponds to solving the following EOV:

EOV: $1000 = 300 a_{\overline{n}|10\%}$

As before we want the OLB at *t*=4. At *t*=4 there 0.2542 payments left. So, we can compute OLB_4 by making one small change in N as shown in Table 9.6B.

N	I	PV	PMT	FV
CPT=4.2542	10	-1000	300	0
0.2542	Keep	CPT=71.80	Keep	Keep

Table 9.6B: C*alculating OLB at t=4. Note the calculator trick. Alternative derivations are given in the text*.

We now reason as in section 9.6:

- I can pay my loan off at *t*=4 with 71.80
- The total payment at *t*=4 is 300+71.80 = 371.80
- If I defer the payment to *t*=5 the debt of 71.80 grows to 71.80 x 1.1 = 71.98.

Suppose you don't like the calculator trick. You can use the retrospective formula to calculate the OLB..

$$OLB_4 = 1000 \times 1.1^4 - 300 s_{\overline{4}|i} = 71.80$$

We then proceed as above

9.8 SOA QIT Problems: Problems on loans, outstanding balances and balloon payments are presented below. Each problem should be worked through until mastery using the concepts presented above.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

Q-IT#16 Annuity - Refinancing - Inflation - Conversion – Patterns; Same as N01#9

Q-IT#64 Refinance Last-payment

Q-IT#75 Refinance

Q-IT#84 Refinance-Last Payment

Q-IT#88 Refinance

Q-IT#89 Refinance (Outstanding problem – uses Cashflows)
QIT#149 Balloon payment

QIT#150 Calculate level payment

QIT#151 Calculate OLB

Q-IT#153 Refinance – 2 Rates

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M00#26 Refinance

M05#8 Refinance

N00#34 Refinance

M00#24 Refinance Deferral

CHAPTER 10

Loans and Sinking Funds

10.1 Overview: Already in Chapter 9 we dealt with loans. If all you are interested in loans is the (level) periodic repayments and refinancing, you can suffice with annuity methods. The methods of this chapter apply when you wish to discuss *interest* and *principle*. Table 10.2 presents a typical amortization table and allows us to discuss interest and principle. In Chapter 10, as in Chapter 9 we deal with a loan of 1000 repaid by five equal end-of-year payments at an effective rate of 10%. The EOV and TV table (Table 10.1) from the previous chapter apply.

EOV: $1000 = Ra_{\overline{5|}10\%}$

N	I	PV	PMT	FV	No Bgin
5	10	-1000	CPT=263.80=R	0	

Table 10.1: *Standard TV line for calculating the level end-of-year payments for a loan.*

t time	R, payments	I, interest	P, principle	OLB, Outstanding loan balance
0				L=1000
1	263.80	10% x 1000=100	263.80-100=163.80	1000-163.80=836.20
2	263.80	10%*836.20=83.62	263.80-83.62=180.18	836.20-180.18=656.02
3	263.80	10%*836.20=65.60	263.80-65.60=198.20	656.02-198.20=457.83
4	263.80	10%*457.83=45.78	263.80-45.78=218.02	239.81
5 =*n*	263.80	10%*239.81=23.98	263.80-23.98=239.81	239.81-239.81=0

Table 10.2a: *Amortization table for the loan presented in Table 10.1*

The basic principles of the amortization table can be seen in Table 10.2

- The payment, *R,* is broken up into two parts, *interest* and *principle.* So 263.80=100+163.80 (Row 1)
- The principle, not the interest, contributes to paying off the loan. The new OLB is the old outstanding loan balance minus the *principle*. It is *not* the old OLB minus the *payment.*
- What is the interest? The interest is a payment to the lender for the privilege of the loan.
- The outstanding loan balance at time *t*=0 is the loan amount, in this case 1000.

- The outstanding loan amount at time $t=n$ is 0, because the loan is paid off.

Table 10.2b illustrates the *Level Principle* method for five payments of equal principle plus interest on a 1000 loan at 10%.

t time	R, payments	I, interest	P, principle	OLB, Outstanding loan balance
0				L=1000
1	200+100	10% x 1000=100	1000/5=200	1000-1*200 = 800
2	200+100-20	10% * 800 =100-20	1000/5=200	1000-2*200 = 600
3	200+100-2*20	10%*600=100-2*20	1000/5=200	1000-3*200 = 400
4	200+100-3*20	10%*400=100-3*20	1000/5=200	1000-4*200 = 200
5 =*n*	200-100-4*20	10%*500=100-4*20	1000/5=200	1000-5*200 = 0

Table 10.2b: Repayment of a loan using the *level principle* method

Table 10.2c illustrates repayment of a loan with arithmetically increasing payments

t time	R, payments	I, interest	P, principle	OLB, Outstanding loan balance
0				L=100v+200v^2= 256.20
1	100	10% x 256.20= 100-74.38 = 25.62	256.20-181.82=74.38	OLB_1 =200v = 181.82
2	200	10%*181.82=200-181.82=18.18	181.82 – 0 = 181.82	0

Table 10.2c: Amortization table for an arithmetically increasing payment loan.

There are about a dozen formulae for loans that govern all loan transactions. The formulae come in groups. There are five formulae governing the *row by row* or *cashflow* amortization table. They are true whether or not the payments are level. They are as follows:

(10.1) $R_t = I_t + P_t$

(10.2) $I_t = i \times OLB_{t-1}$

(10.3) $OLB_t = OLB_{t-1} - P_t$

(10.4) $OLB_0 = L$

(10.5) $OLB_n = 0$

10.2 The Four Outstanding Loan Balance Formulae: Frequently, the most important step in a problem is finding the immediately preceding OLB. The reason for this is that every problem shifts the 0 point and the OLB is the most convenient way of shifting the 0 point to where you are. The four OLB formulae were already stated in Chapter 9. They are repeated here because of their importance. These formulae are true independent of whether payments are level or not.

(10.6) $OLB_t = \text{PV(All future payments)}$ e.g. if payments are level $OLB_t = Ra_{\overline{n-t}|}$

(10.7) $OLB_t = L(1+i)^t - \text{AV(All previous payments)}$ e.g. if payments are level $OLB_t = L(1+i)^t - Rs_{\overline{t}|}$

(10.3) $OLB_t = OLB_{t-1} - P_t$

(10.8) $OLB_t = \text{PV(Next } s \text{ payments)} + OLB_{t+s}v^s$, e.g. if payments are level $OLB_t = Ra_{\overline{s}|} + OLB_{t+s}v^s$

In Chapter 9, we have pointed out that each of these formulae has a place and use:

- You use (10.6) when *n* is known
- You use (10.7) when *n* is unknown but *t* is known
- You use (10.8) when *n* and *t* are unknown
- You use (10.3) when you desire to calculate P_t

10.3 The Two Level Formula: Frequently, payments are level. In addition to the previous formulae we can use the following two formulae

(10.9) $L = Ra_{\overline{n}|}$, $OLB_t = Ra_{\overline{n-t}|}$

(10.10) $P_t = Rv^{n+1-t}$

The derivation of (10.10) is an instructive exercise in timelines and is presented in Table 10.3

Time	0	1	…	*n-t*	*n*-(*t*-*1*)	
t		R	R…	R		$OLB_t = Ra_{\overline{n-t}|}$
t - 1		R	R…	R	R	$OLB_{t-1} = Ra_{\overline{n-(t-1)}|}$
Subtract!	0	0	0…	0	R	$PV = Rv^{n-(t-1)}$

Table 10.3: *Derivation of (10.10).*

Equation (10.10) is a remarkable formula. It says that the principle payments form a geometric progression. This allows powerful problems requesting construction of the entire loan transaction often from almost nothing. The following problem is illustrative.

Remarkable problem: For a loan of 1000, with level payments, P_2 = 180.18, P_4 = 218.02, Construct the entire amortization table.

Notice what the problem does not give you. You are neither told *R*, nor *i*, nor even *n*. The problem gives an appearance of impossibility, the need to construct information that is not there. Here is a bulleted statement of the solution.

- Using (10.10) we have 218.02/180.18 = $(1+i)^2$ ➔ 1+*i* =1.1.
- But then $P_1 = P_2/1.1$ = 180.18/1.1 =163.80
- $I_1 = i \times L$ = 10% x 1000 =100
- $R = P + I$ = 100+163.80 = 263.80

- $v^n = P_1/R$ = 163.80/263.80 = 0.6179 ➔ n = 5
- Since L=1000, n=5, and i=10 we can construct the entire amortization table, 10.1. Amazing☺

10.4 The Three Total Formulae: There are 3 total formulae. An important English point should be made: Normally, throughout this course, when we ask for the value of a collection of payments we refer to their actuarial value. In loans the phrases *total payments, total principle,* or *total interest* refer to raw sum totals without any accumulation of interest. There are legal reasons for doing this; we want lenders for example to be aware of how much they are spending in dollars over the course of the loan.

$$(10.11) \qquad \text{Total (level) payments } = \text{Raw some (no discount) of all payments} = nR$$

It should be obvious how to modify (10.11) for non-level payments are for a portion of the payments (e.g. total of last 5 payments).

Again under assumption of level payments we have

$$(10.12) \qquad \text{Total principle} = \text{Raw sum (no discount) of all principle payments} = L = Rv^n + Rv^{n-1} \ldots Rv = Ra_{\overline{n}|i}$$

More generally we can sum principles at $t=m,\ t=m+1, t=m+2,\ldots,t=m+s-1$, under assumption of level payments.

t
$$(10.13) \qquad \text{Sum of principles at times } t = m, m+1, \ldots, m+s-1 = Rv^{n-m}a_{\overline{s}|}$$

The derivation of (10.13) is elegantly demonstrated using a timeline as seen

t	0	1…	*n+1-m*	*n+1-m+1*	…	*n+1-m+s-1*	
Cashflow			1	1	…	1	
PV			v^{n+1-m}	$v^{n+1-m+1}$	…	$v^{n+1-m+s-1}$	
Count			*1st*	*2nd*		*s-th*	

Table 10.4: *Derivation of (10.13)*

Here is the point: By (10.10), $P_m = R\,v^{n+1-m}$. But v^{n+1-m} is the PV of a payment of 1 at $n+1-m$. Since there are a total of s payments the result is R times a deferral factor times an s-payment annuity.

We next present formulas and examples for payment by level principal (10.2b)

$$(10.14) \qquad P_t = \text{Principal at time } t = \frac{L}{n}$$

$$(10.15) \qquad OLB_t = \text{Outstanding loan balance at time } t = L - \frac{L}{n} \times t$$

$$(10.16) \qquad I_t = i \times L - i \times (t-1) \times \frac{L}{n}$$

To derive (10.16) simply multiply the interest rate i by OLB_{t-1} found in (10.15).

When summing (10.16) over many periods the following formulas are useful

$$(10.17) \qquad 1+2+\ldots+n = \frac{n(n+1)}{2}, \qquad m+m+1+\ldots+n-1+n = \frac{n(n+1)}{2} - \frac{m(m-1)}{2}$$

Example: For a $4800 loan at ½% per period with level principal payments paid over 4 years with monthly payments what is the total payment in 2nd year

Solution:

- By (10.1), Total Payment = Total P_t + Total I_t
- By (10.14), P_t = 4800/48 = 100
- By the previous two bullets, the *Total Principal* is 12 months x 100 = $1200
- The 1st year ends at t=12. The 2nd year begins at t=13, and ends at t=24.
- By (10.16), I_{13} = ½% x 4800 – ½% x 12 x 100 = 24 -12 x 0.5; I_{14}= 24- 13 x 0.5, …I_{24}=24-23 x 0.5
- So Total interest in 2nd year is 12 x 100 – 0.5 x (12+13+…+23)
- By (10.17) 12+13+…+23 = 210. One can also enter it manually on calculator (Not so bad)
- Hence *Total interest* = 12 x 24 – 0.5 x 210 = 183
- Hence *Total Payment = Total Principal + Total Interest* = 12 x 100 + 12 x 24 – 0.5 x 210 = 1383

10.5 Calculator Spreadsheet Did you know that the BA-II has spreadsheet capacity? To see this enter the keystrokes in Table 10.1 the basic loan calculation. Now hit `2nd Amort` which is above the PV key. Here are some questions you can answer. This is equivalent to creating a spreadsheet with the entire loan and *seeing* it.

Question: What is I_3, P_3, OLB_3?

Answer: `Scroll down (down arrow) until you see P1. Hit 3 ENTER`

`Scroll down 1 and  you will see P2. Hit 3 ENTER`

```
Scroll down 1: OLB3 = 457.83
Scroll down 1: P3 = 198.20
Scroll down 1: I3 = 65.60
```

Explanation: P1 and P2 give you a window *from* row 3 *to* row 3. In that window you can *see* the spreadsheet and you can see OLB_3, P_3, I_3.

Question: What is the total interest paid in the first two periods. What is the total principle paid in the first two periods.

Answer: `Scroll down (down arrow) until you see P1. Hit 1 ENTER`

`Scroll down 1 and  you will see P2. Hit 2 ENTER`

```
Scroll down 1: OLB2 = 656.03
Scroll down 1: P1 + P2 = 343.97
Scroll down 1: I1 + I2 = 183.62
```

Explanation: P1 and P2 give you a window *from* row 1 *to* row 2. In that window you can *see* the spreadsheet and you can obtain the snapshot given above.

Question: What is the total interest paid in the first two periods. What is the total principle paid in the first two periods.

Answer: `Scroll down (down arrow) until you see P1. Hit 1 ENTER`

```
Scroll down 1 and  you will see P2. Hit 5 ENTER

Scroll down 1: OLB5 = 0
Scroll down 1: P1 +…+ P5 = Total Loan = 1000
Scroll down 1: I1 +…+ I5 = 318.99 = Total Interest
```

Explanation: P1 and P2 give you a window *from* row 1 *to* row 5. In that window you can *see* the spreadsheet and you can obtain the snapshot given above.

Cautions: Please avoid the following two calculation errors

- Hit P1 before P2 (Otherwise you get ERROR 3. To fix ERROR 3, turn off the calculator, reenter amortization worksheet and hit P1 before P2)
- By ENTER I mean the ENTER key (row 1, column 2) not the = key.

10.6 Sinking Funds: A sinking fund is an alternate way to pay a loan. The basic idea is to pay level interest payments to the loaner. As to the loan, the lender makes deposits into a fund to accumulate the exact loan amount without *further* accumulation of interest and pay back the loaner at a mutually agreed date.

We can construct the following example. A person makes a 1000 loan to be paid back in 5 years. For the privilege of receiving 1000 the lender must pay 15% to the loaner. The lender will accumulate the 1000 in a five year account which accumulates at 7%. The results are compactly exhibited in Table 10.5.

t	Interest $= i_L \times L$ = 15% x 1000	$Ds_{\overline{5}\|7\%} = 1000$	I^{SF}_t;Interest on SF; $i_{SF} \times SF_{t-1}$	Sinking fund net amount, SF_t	OLB_t, Balance on loan @ t		
0				0	1000-0=1000		

1	15%(1000)=150	173.89	7% x0=0	$173.89 = Ds_{\overline{1}\rceil 7\%}$	1000-173.89=826.11		
2	15%(1000)=150	173.89	7%x173.89=12.17	359.95	1000-359.95=640.05		
3	15%(1000)=150	173.89	7%x359.95=25.20	559.04=359.95+173.89+25.20	1000-559.04=440.96		
4	15%(1000)=150	173.89	7%x559.04=39.13	$772.06 = Ds_{\overline{4}\rceil 7\%}$	1000-772.06=227.94		
5	15%(1000)=150	173.89	7%x772.06=54.04	1000=772.06+173.89+54.04	1000-1000=0		

Table 10.5: *A sinking fund for a 1000 loan. There is further elaboration in the text.*

The concepts emerging from Table 10.5 should be clear

- There are two interest rates: The rate to the loaner, i_L and the rate in the sinking fund, i_{SF}
- *D* is called the *sinking fund deposit.* Its equation of value is a simple annuity equation as shown
- Column 4 gives the interest on the SF given by the bank
- Column 5 gives the *net sinking fund amount.* Note the two formulae: You can use an annuity formula *or* you can use a cashflow approach where the net amount is the sum of i) previous amount ii) interest on previous amount and iii) *D*, the deposit.
- Column 6 gives the *outstanding balance.* Since the loan is not accumulating interest (Why? Because the interest is being paid separately) at any point the outstanding loan is *L*=1000 minus the sinking fund amount.

10.7 Sinking Fund English: I don't like sinking fund problems. They are not usually very deep and only test knowledge of special English phrases. I summarize these phrases in Table 10.6

English Phrase	Formula	Example from Table 10.5
Total payment	i_L x $L + D$	15%(1000)+173.89=323.89
Net interest payment @ *t*	i_L x L - I^{SF}_t	15%(1000)-25.20=124.80 @ *t*=3
Net Balance on the loan @ *t*	OLB_t	Net balance on the loan @ time *t*=3 is 440.96

Table 10.6: *Three important phrases connected with Sinking Funds and what they mean.*

10.8 SOA QIT Problems: Problems on loans, involving interest and principle are presented below. Each problem should be worked through until mastery using the concepts presented above.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

Q-IT4 Sinking-fund Amortization

Q-IT15 Amortization - Examples - 13 Basic Formulae - Comparison - Total Formulae; Same as N01#6

Q-IT24 Loan - Amortization - Sinking Fund; Same as M01#4

Q-IT26 Amortization - Money Growth - 13 Formulae - English

Q-IT28 Amortization - 13 Formulae

Q-IT46 Loans - 13 Formulae

QIT63 Loans - 13 Formulae - Total Formulae

Q-IT80 Loans - Amortization Table - Sinking Fund

QIT81 5 Level Formulae-5 General Formulae

QIT86 5 Level Formula-5 General Formulae

Q-IT87 Sinking Fund - 2 interest rates

Q-IT89-Outstanding problem – Cashflow approach

QIT148 – Payment = Interest (Not posted yet)
QIT152 – Principle trick – Balloon payment (Not posted yet)

FROM THE ARCHIVED EXAMS: No solution posted.
SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M05#2 Sinking Fund

N00#48 Sinking Fund

M03#15 Sinking-fund Amortization

M01#7 4 Total Formulae

N00#55 4 Total Formulae

M01#13 5 Level Formulae

M01#37 5 Level Formulae

N05#18 5 Level Formulae

M03#39 5 Level Formulae - 5 General Formulae

N00#12 5 Level formulae - 3 Total Formulae

CHAPTER 12

Bonds

12.1 Overview: What is a *bond*? A bond is simply an IOU, a statement of debt. It typically arises in the following manner:

> Suppose I wanted to create the Dr. Hendel Pen company. I might *initially* need 1,000,000 to buy pen making equipment, to rent property and to hire staff. A bank would not loan me 1,000,000 since I have no proof the pen company would work; if it didn't work I could declare bankruptcy and the bank would lose the 1,000,000 it loaned me.
>
> However, another way to raise the 1,000,000 is to issue bonds. Suppose I issue 1000 bonds. Each bond would cost the purchaser 1000. So I would raise 1000 x 1000 = 1,000,000.
>
> - Each owner of the bond would be entitled to *redeem* the bond at *maturity* at time *n.* At time *n,* I, the bond issuer, would pay each purchaser the redemption value of the bond. For example the redemption value of the bond might be 1500. In this way, each bond purchaser makes money.
> - Each owner of the bond might be entitled to receive *coupons* of the bond. A coupon simply means that at periodic intervals, I, the bond issuer, pay a certain amount.
>
> So the bond purchaser pays 1000 and in turn receives the redemption value and coupons if they are offered. If the bond pays no coupons it is called a *zero-coupon* bond.

There are rating agencies that rate bonds. Typical ratings are AAA, AA, A, BBB, BB, B, C. A good rating means the rating agency expects the bond issuer to make all payments. A low rating, say C, indicates that the rating agency considers the bond risky. The bond issuer may default, declare bankruptcy, and the bond purchaser would not be able to collect coupons or a redemption value. In such a case the purchaser of the bond is at a loss. You can look up bond ratings on the internet. There are several rating agencies.

12.2 Notations, Conventions, and Special English Phrases: The following terms, conventions and special English phrases are used in connection with bonds.

- P is the price the bond purchaser pays for the bond
- The *term* of the bond or the *maturity date* or the *redemption* date indicates when the bond is redeemed.

- If payments are annual then *n* indicates the *term* of the bond, *the maturity date,* as well as the number of payment periods. However, if coupons are paid several times a year, then *n* must be multiplied to obtain the number of *payment periods*.
- *C* is the *redemption value* of the bond which the bond issuer pays the bond investor at maturity
- *F* is called the *face value* of the bond; *r* is called the *coupon rate* of the bond. The product *Fr* is the coupon amount. This is a bit spooky since *F* and *r* are only used to obtain their product, the coupon amount.
- *r* is always interpreted nominally even if you are not explicitly told so. So for example, a coupon rate of 4% on a bond that pays twice a year would mean a coupon rate of 2% each half-year. Note, although *r* is always nominal, the interest rate for the bond is only nominal if you are explicitly told so.
- Frequently, *C=F.* We say that such a bond is redeemed *at par.* Alternatively, we might say that the buyer purchased a *par-value bond.* If you are not told anything you can assume *C=F.* But if a problem indicates that *C* and *F* might be different than you cannot assume *C = F.*
- *i* is the annual effective rate of the bond. It is the rate at which the present value of outflow, the price, equals the present value of all inflows, the coupons and redemption price. If the bond makes payment several times a year you must take roots of the interest factor to get the interest rate per period. You cannot treat the interest rate nominally unless you are explicitly told so. Contrastively, the coupon rate *r* is always nominal even if you are not told so.
- We define the symbols *g* and *G,* such that *Fr = Cg = Gi.* In a certain sense *g* and *G* are fictitious. They indicate alternate ways of calculating the coupon amount. They are not really used for anything else.

12.3 Bond Timelines and Formulae: The typical bond timeline is presented in Table 12.1

Time	0	1	2	…	*n*
Cashflow	*P*	*Fr*	*Fr*	….	*Fr, C*

Table 12.1: *Typical timeline for a bond that makes coupon payments once a year*.

The equation of value is given by (12.2) and follows known principles.

(12.2) **Basic formula** $P = Fra_{\overline{n}|i} + Cv^n$

The typical TV line for a bond is given in Table 12.3 below.

N	I	PV	PMT	FV
n	*i*	*-P*	*Fr*	*C*

Table 12.3: *Typical TV line for a bond. Inputting any 4 quantities would allow computation of the* 5^{th}.

The following illustrative example will illustrate how to deal with payments several times a year.

Illustrative Example: Find the annual effective yield on a 1000, 10 year, par-value bond with coupon rate of 4% which makes coupon payments twice a year and is purchased for 1100.

Illustrative Solution: *Par value* means that $C=F$. Since coupons payments are made twice a year, we have 20 payments and the coupon rate per payment period is 2%. Hence the coupon amount is $Fr = Cr = 1000 \times 2\% = 20$. Table 12.4 presents the timeline. Table 12.5 presents the TV line. The equation of value is derived from (12.2) with appropriate numbers plugged in. The equation of value is

$$1100 = 20a_{\overline{20|}j} + 1000Cv_j^{20}$$

Time	0	1	2	…	20
Cashflow	*-1100*	*20*	*20*	….	*20, 1000*

Table 12.4: *Timeline for illustrative example*.

N	I	PV	PMT	FV
20	*CPT=1.4220%*	*-1100*	*20*	*1000*

Table 12.5: *TV line for illustrative example.*

1.4220% is the interest rate per payment period, which is a half-year. The interest rate per period is obtained by squaring the appropriate factor. $1.014220^2 = 1.0286$ ➔ $i = 2.86\%$ annual effective rate.

It is instructive to *understand* what this means. That is, with all these payments – purchase of 1100, redemption of 1000 and coupons of 20 – what does it mean to say that the effective rate per period is 1.422%. The meaning of this rate can be seen by a cashflow table. At time 0 you place 1100 in an account earning 1.4220% per half year. Then at time t=1/2 the account has 1100 x 1.014220 =1115.64. But at time t=1/2 there is a coupon payment of 20 so that the net amount in the account at time t=1/2 is 1095.64. This is best presented in a spreadsheet which is presented in Table 12.6. At time t=10, the 20th period, the account has 1000 left which is used to pay the redemption value. Upon payment the account is closed without leftover amount.

12.4 Bond Amortization Table, Bond Formulae: The *amortization* table from Chapter 10, Table 10.2 still holds for bonds. The various amortization formulae hold with some changes. First there are important nomenclature changes which are summarized in Table 12.7

For loans, we refer to …	**For bonds, we refer to…**
Loan amount, *L*	Price, *P*
Periodic payments, *R*	Coupons *Fr*
Interest, I_t	Interest, I_t
Outstanding loan balance, OLB_t,	Book Value, BV_t
Principle, *P*	Principle, also called *adjustment to principle*

Table 12.7: *Difference in terminology between loans and bonds.*

Specifications:	P=1100, r=2% @ half year, C= 1000, i=1.4220%(@half year)		
Time \ Amounts	Begin Amount	End Amount	Comment
0	1100.00	1100.00	*Bond purchased for 1100*
1	1115.64	1095.64	*Interest of 1.422% on 1100*
2	1111.22	1091.22	*Subtract 20 from account (Coupon)*
3	1106.74	1086.74	
4	1102.19	1082.19	
5	1097.58	1077.58	
6	1092.90	1072.90	
7	1088.16	1068.16	
8	1083.35	1063.35	
9	1078.47	1058.47	
10	1073.52	1053.52	
11	1068.50	1048.50	
12	1063.41	1043.41	
13	1058.25	1038.25	
14	1053.01	1033.01	
15	1047.70	1027.70	
16	1042.32	1022.32	
17	1036.86	1016.86	
18	1031.32	1011.32	
19	1025.70	1005.70	

20	1020.00	1000.00	*1000 left for redemption value*

Table 12.6: *Cashflow diagram for illustrative example.*

Instead of (12.2) we can use the following formula to calculate price.

(12.8) Premium – Discount Formula $P = (Fr - Ci)a_{\overline{n}|i} + C = C(g-i)a_{\overline{n}|i} + C$

This important formula can be derived by substituting $v^n = 1 - ia_{\overline{n}|i}$ into (12.2). Using the notation and terminology adjustments from Table 12.7, we now list the formulae from Chapter 10 indicating where changes take place.

(12.9) $Fr = I_t + P_t$

(12.10) $I_t = i \times BV_{t-1}$

(12.11) $BV_t = BV_{t-1} - P_t$

(12.12) $BV_0 = P$

(12.13) $BV_n = C$, this formula changed; BV_n is not 0

(12.14) $BV_t = Fra_{\overline{n-t}|i} + Cv^{n-t} = C(g-i)a_{\overline{n-t}|i} + C$; this formula changed, but the concept, FV of future payments, is same

(12.15) $P_t = C(g-i)v^{n+1-t}$, this formula changed but the principle payments still form a geometric progression

12.5 Premium / Discount: If a bond is zero coupon and if you paid more than the redemption value, *P >C*, then you would lose money. But if you get coupons then it is possible that *P>C*. It is also possible that P < C whether or not you pay coupons.

Recall from (12.11) that book values decrease by P_t. But if P_t is negative then the book values *increase*. This can happen when *C>P*. *P* corresponds to the book value at time 0. That book value must *increase* to *C* in order to make the redemption payment. This gives rise to a variety of terminologies to deal with the negative principle payments. These differences are summarized in Table 12.16. The terminology is used so that certain negative values can be spoken of positively. It can be tricky in word problems if a negative number is phrased as a positive number.

Comparison of price and redemption value	*P > C*, (*g > i*)	*P < C*, (*g < i*)
Terminology for purchase	Bond bought at premium	Bond bought at discount
Description of difference (always positive)_	Amount of premium = *P-C*	Amount of discount = *C-P*

Sign of principle	Positive	Negative
English description of principle (always positive)	Amount of amortization of premium in *t*-th coupon is P_t	Amount of amortization of discount in *t*-th coupon is $-P_t$
Alternative description of principle (always positive)	Amount of write-down in t-th coupon is P_t	Amount of write-up in *t*-th coupon is P_t
Principle	Amount of adjustment of principle	Amount of adjustment of principle

Table 12.16: *Terminology depending on whether bond is bought at premium or discount.*

12.6 Book Value between Coupon Dates: The problem with finding a book value between coupon dates is that if you just multiply the book value at the previous coupon date by $(1+i)^f$ where f corresponds to the fraction of period involved and i corresponds to the rate per payment period, then you forget about the coupon! So if you take $(1+i)\ BV_t$ you do not get BV_{t+1} since $(1+i)\ BV_t = BV_{t+1} - Fr$. How do you deal with the coupon? Do you ignore it? Subtract a portion of it? It turns out there are four approaches to dealing with book values between coupon dates. They are presented in Table 12.17. Table 12.18 presents the TV line for calculating the *theoretical clean* value.

Name	Theoretical	Practical (linear approximation to theoretical or Bernoulli)	Reason for name
Dirty	$BV_t(1+i)^f$	$BV_t(1+fi)$	Graph of BV_t is *dirty*. It has *jumps* on every coupon date because coupon was not dealt with till end
Clean	$BV_t(1+i)^f - Cgs_{\overline{f}\vert i}$	$BV_t(1+fi) - Cgf$	Graph is connected

Table 12.17: *Formulas for calculating BV_{t+f} where t is an integer, f is a fraction and i is rate per period.*

N	I	PV	PMT	FV
$n-(t+f)$	i	CPT	Fr	C

Table 12.18: *The above TV line calculates the theoretical book value at t+f if the bond has n payment periods.*

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#10 Bonds - 13 Bond Formulae - Plugin - English Conventions

QIT#22 Bond - Algebra - English

QIT#62 13 Bond Formula Formulae

QIT#74 Comparison

QIT#76 Comparison

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

N05#4 Plug in (Basic Formula)

N05#24 Plug in (Basic Formula)

M00#29 Refinancing

M03#42 8 Formulae

N01#31 Comparison

M05#5 Comparison

CHAPTER 13

ReInvestment

13.1 Overview: What is *reinvestment?* The simplest example of reinvestment is when you purchase a bond at one interest rate and then reinvest the coupons at another interest rate. This raises the question as to what the overall yield is. The following problem is illustrative.

Illustrative problem: *An investor purchases a 6%, 1000, 5-year, par value bond with semiannual coupons yielding 8% convertible twice a year. Immediately upon receiving each coupon, the investor reinvests the coupon in an account earning 10% convertible twice a year. Calculate the overall yield to the investor.*

13.2 The Basic Three Principles by Which to Solve ReInvestment Problems: All reinvestment problems may be solved by applying the three principles listed in Table 13.1. Note, reinvestment problems come in a variety of flavors and forms; therefore, there is no one formula that solves all reinvestment problems. However, there is one method, the method in Table 13.1, which always works.

Principle	Description
1. One Timeline for each Interest rate	Think of each interest rate as coming from a separate bank. Each interest rate, or each bank, then gets its own timeline.
2. Timeline EOV	Make sure each timeline has a timeline EOV relating beginning or ending values with some actuarial function
3. Summary Line	To solve the problem you must create a *Summary Timeline.* The summary timeline must contain all *inflows and outflows.* The equation of value relating these inflows and outflows is the key to solving the reinvestment problem. This is the hard part of any reinvestment problem.

Table 13.1: *The three principles necessary to solve any reinvestment problem.*

13.3: Timelines for the Illustratative Problem: Table 13.2 applies Table 13.1 to the illustrative problem.

Timeline	**0**	**1**	**2**	**…**	**10**	**EOV**
Timeline 1	-P	30	30	…	30,1000	$P = 30a_{\overline{10}\|4\%} + 1000v_{4\%}^{10} = 918.89$
Timeline 2		30	30	…	30	$A_2(10) = 30s_{\overline{10}\|5\%} = 377.34$

Summary Timeline	*Outflow*, *P*=918.89				*Inflow: 1000 377.34*	$P(1+i)^{10} = 1377.34$, rate factor per year is $(1+i)^2$ $i = 4.13\%$; $1.0413^2 = 1.0843$; rate per year $= 8.43\%$

Table 13.2: *Application of the reinvestment rules of Table 13.1 to the illustrative problem.*

13.4 Subtleties: In solving the illustrative problem using the three-step method of Table 13.1, subtleties arise.

- Notice how *i* is not used until the summary line. The other timelines *each* have their *own* interest rate
- The sole purpose of the 4% is to calculate *P.* It is not needed otherwise
- Notice how there is one inflow *per* timeline. Timeline 1 has an inflow of 1000; Timeline 2 has 377.34
- The EOV for timeline 2 is not obvious. Think of timeline 2 as another bank. This other bank gives 377.34.
- In the above problem the summary timeline EOV is a chapter 2 problem. More complicated patterns exist.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#7 - Reinvestment - Increasing – Examples
QIT#47 – Reinvestment

QIT#114- Reinvestment

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M01#41 Reinvestment

N05#11 Reinvestment

N05#16 Reinvestment

CHAPTER 13

ReInvestment

13.1 Overview: What is *reinvestment?* The simplest example of reinvestment is when you purchase a bond at one interest rate and then reinvest the coupons at another interest rate. This raises the question as to what the overall yield is. The following problem is illustrative.

Illustrative problem: *An investor purchases a 6%, 1000, 5-year, par value bond with semiannual coupons yielding 8% convertible twice a year. Immediately upon receiving each coupon, the investor reinvests the coupon in an account earning 10% convertible twice a year. Calculate the overall yield to the investor.*

13.2 The Basic Three Principles by Which to Solve ReInvestment Problems: All reinvestment problems may be solved by applying the three principles listed in Table 13.1. Note, reinvestment problems come in a variety of flavors and forms; therefore, there is no one formula that solves all reinvestment problems. However, there is one method, the method in Table 13.1, which always works.

Principle	Description
4. One Timeline for each Interest rate	Think of each interest rate as coming from a separate bank. Each interest rate, or each bank, then gets its own timeline.
5. Timeline EOV	Make sure each timeline has a timeline EOV relating beginning or ending values with some actuarial function
6. Summary Line	To solve the problem you must create a *Summary Timeline.* The summary timeline must contain all *inflows and outflows.* The equation of value relating these inflows and outflows is the key to solving the reinvestment problem. This is the hard part of any reinvestment problem.

Table 13.1: *The three principles necessary to solve any reinvestment problem.*

13.3: Timelines for the Illustratative Problem: Table 13.2 applies Table 13.1 to the illustrative problem.

Timeline	0	1	2	…	10	EOV
Timeline 1	-P	30	30	…	30,1000	$P = 30a_{\overline{10}\|4\%} + 1000v_{4\%}^{10} = 918.89$
Timeline 2		30	30	…	30	$A_2(10) = 30s_{\overline{10}\|5\%} = 377.34$

Summary Timeline	*Outflow, P*=918.89				*Inflow: 1000 377.34*	$P(1+i)^{10} = 1377.34$, rate factor per year is $(1+i)^2$ $i = 4.13\%$; $1.0413^2 = 1.0843$; rate per year $= 8.43\%$

Table 13.2: *Application of the reinvestment rules of Table 13.1 to the illustrative problem.*

13.4 Subtleties: In solving the illustrative problem using the three-step method of Table 13.1, subtleties arise.

- Notice how *i* is not used until the summary line. The other timelines *each* have their *own* interest rate
- The sole purpose of the 4% is to calculate *P.* It is not needed otherwise
- Notice how there is one inflow *per* timeline. Timeline 1 has an inflow of 1000; Timeline 2 has 377.34
- The EOV for timeline 2 is not obvious. Think of timeline 2 as another bank. This other bank gives 377.34.
- In the above problem the summary timeline EOV is a chapter 2 problem. More complicated patterns exist.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#7 - Reinvestment - Increasing – Examples
QIT#47 – Reinvestment

QIT#114- Reinvestment

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M01#41 Reinvestment

N05#11 Reinvestment

N05#16 Reinvestment

CHAPTER 14

IRR, Dollar, Time Weighted, Stock

14.1 Overview: We have a collection of small topics to cover. Each topic is governed by a single formula or method. The topic is small in the sense that it is completely covered by that single formula or method. That is all you have to learn. We could have a chapter for each one, but the topics are so small it makes more sense to cover them together.

14.2 Internal Rate of Return, IRR: Consider the following problem which we will analyze in two ways.

Illustrative problem: *An account has 1,000 on January 1st. 300 is deposited on April 1st and 400 is withdrawn on September 1st. On December 31st the account is worth 950. Calculate the annual effective yield.*

This problem could be typical in a bank or other investment account. Money is continually going in and out. We can set the problem up by traditional means. The timeline is presented in Table 14.1 and the EOV in (14.2).

Time	0	3 months	8 months	12 months
Time	0	¼	2/3	1
Cashflow	1000	300	-400	-950

Table 14.1: *Cashflows in an investment account. Note the use of month-year and absolute time. Throughout this chapter cashflow refers to either deposits or withdrawals.*

(14.2) $$1000+300v_i^{\frac{1}{4}}-400v_i^{\frac{2}{3}} = 950v, \qquad i \text{ is the annual effective rate}$$

Here is the problem. Equation (14.2) can't be solved algebraically. For example, if you substitute $X^{12}=v$ in (14.2) you obtain an equation of degree 12 which cannot be solved algebraically with a closed formulae.

To solve this problem, we can use the `CashFlow` sheet of the BA II calculator. To enter it hit the `CF` button on row 2 column 2 of the BA II Plus. You might want to clear the `CashFlow` memory by hitting the following keys: `2nd (`Col 1, Row 2`) CLR WORK (`Col 1, Last row`) 2nd SET (`Row 1, Col 2`) 2nd QUIT (`Row 1, Col 1`)`. You can *navigate* the `CashFlow` worksheet using the `up` and `down` arrows found in the 1st row. When you wish to enter a number you simply key in the number and hit `ENTER` (i.e., =). Because only integer entries of time are allowed we key in by month, solve for the monthly effective yield and then convert to an annual effective yield. The entries into the `CashFlow` sheet are presented in Table 14.3.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
CF Sheet	CFo	C01	C02	C03	C04	C05	C06	C07	C08	C09	C10	C11	C12
Entry	1000	0	0	300	0	0	0	0	-400	0	0	0	-950

Table 14.3: *Entries into the CashFlow sheet corresponding to the cashflows in Table 14.1.*

To compute the effective yield per period we hit the keys `IRR CPT`, and obtain 0.37%. We can then calculate the annual yield using the equation $1.0037^{12} = 1.0458$.

As a general rule of thumb, use the `IRR` whenever i) you wish to calculate *i*, and (ii) the EOV for the problem cannot be solved algebraically.

14.3 Net Present Value, NPV: In the previous problem you were given some *inflows* and *outflows* and asked to calculate the *interest rate.* Sometimes the complement is done; you are given an *interest rate* and *outflows* and asked to calculate the Net Present Value. This too can be done with the `CashFlow` sheet. We modify the above example to illustrate

Example: 1000 is deposited in an account with a 300 deposit at *t*=3, a 400 withdrawal at *t*=8, and an ending balance of 950. The account pays 0.5% per month. What is the Net Present Value (NPV) of this project.

Solution: The timeline is given in Table 14.1; the EOV is given by equation (14.4).

$$(\mathbf{14.4}) \qquad \mathbf{NPV} = -1000 - 300v_i^{\frac{1}{4}} + 400v_i^{\frac{2}{3}} + 950v_i$$

One can simply calculate this using calculator strokes. However, one can use the NPV feature of the BA II. This is illustrated in Table 14.3. You input the *same* values that you would input for IRR.

To find the `NPV`, you hit the `NPV` button and when prompted for `I,` input 0.5 (for 0.5%). You then scroll, using the up and down keys, to the `NPV` window and hit compute `CPT`. You should get -16.3805.

14.4 Dollar Weighted Approximation: Today we have calculators that can quickly solve EOVs like (14.2). Prior to calculators various *approximations* were used. Two very popular approximations are the Time Weighted and Dollar Weighted approximation. The Dollar Weighted approximation is also called the Money Weighted approximation. Dollar weighted approximation starts off with Table 14.1 and (14.2). Equation (14.5) shows how the EOV is approximated so that one can solve for *i*.

(**14.5**) EOV	$1000 + 300v^{\frac{1}{4}} - 400v^{\frac{2}{3}}$	$=950v$
multiply by $1 + i$	$1000(1+i) + 300(1+i)^{\frac{3}{4}} - 400(1+i)^{\frac{1}{3}}$	$=950$
Use Bernoulli, $(1+i)^f \approx 1 + fi$	$1000(1+i) + 300(1 + \frac{3}{4}i) - 400(1 + \frac{1}{3}i)$	$=950$
Simplification	$1091\frac{1}{6}i$	$=50$
Solve linear equation for i	i	$=4.58\%$

Three important points should be emphasized:

- IRR vs Dollar weighted: In this example, the dollar weighted yield equals the exact IRR. This is rare. Typically the two rates differ. Sometimes the dollar weighted rate is more and sometimes it is less than the exact yield.
- EOV at t=0 and t=1: The EOV in (14.5) equates PV at 0 and then multiplies by (1+i) which in effect equates current values at 1. Paradoxically, if the Bernoulli approximation is applied to the original EOV of PV at 0, one obtains a different answer. It it important to equate current values at 1.
- One Period: In this example, the investments took place over a year. When we labeled the times we called the end of the period 1. Even if the investments took place over several years, the end of the time period is always called one. That is, the method calculates the yield *for the period of investment.* If one then wants an annual effective yield, one must take appropriate roots of factors.
- Focus of dollar weighted: In the dollar weighted method:
    - The following items are needed (if not present create variables for them):
        - Beginning and ending account balances,
        - cashflows (deposits and withdrawals);
        - The times of the Beginning (t=0), Ending (t=1) balances, and cashflows
    - The following items are not needed:
        - Intermediate account balances (i.e. If a cashflow is made at time t, how much was the account worth at time t, prior to cashflow, and how much is it worth at time t, after the cashflow)
- *Exposure, Interest*: The 50 on the right hand side of (14.5) represents the excess funds at the end of the investment period over the raw amounts of cashflows during the year; hence the 50 represents *I = Interest amount.* The 1091.1666i on the left-hand side of (14.5) represents *exposure.* To understand this, note that the initial balance amount, 1000, is *exposed* to the interest rate in the account for the entire investment period; contrastively, the 300 deposit is *exposed* to the interest rate in the account for only the last ¾ of the investment period (since it wasn't there at the beginning of the investment period).

14.5: Time Weighted Approximation: Table 14.1 does not present sufficient information to evaluate the Time Weighted approximation. The Time Weighted approximation requires knowledge of the beginning balances just prior to any cashflows. Thus, we extend the illustrative example with the following information.

Illustrative example extension: *In addition to the statement in the illustrative problem, we further assume that the account balance on April 1 and September 1st just prior to the cashflows are 1200 and 1700 respectively. Although we retain the assumption of a beginning balance of 1000 we assume the end balance on December 31st is 1500 (not 950). Table 14.6 presents the computations for the Time Weighted approximation.*

Time (Ignore)	0	¼	2/3	1
Beginning balance	1000	*1200*	1700	**1500**
Cashflow	NA	+300	-400	NA
End Amount	*1000*	1500	**1300**	1500
Rate factor from last cashflow to present	NA	1200/1000=1.2	1700/1500=1.1333	1500/1300=1.1538

Table 14.6: *Table template for the time weighted approximation:* $1+i_{TimeWeighted}=1.2 \times 1.1333 \times 1.1538=1.5692$

We may summarize the method as follows:

- The following information is needed (if not present create variables for them):
  - Beginning and end balance (for entire period),
  - Beginning balance just prior to each cashflow.
  - The end balance just after each cashflow is computed.
- The following information is not needed:
  - The dates (or time) at which cashflows are made

The time-weighted factor ($1+i_T$) is computed as follows:

- At the time of *each* cashflow a yield from the last cashflow (or beginning period) to the current cashflow (or end amount) is computed by dividing current amount over previous amount. This gives the *rate factor* corresponding to each cashflow.
- The product of rate factors equals (by definition) the Time-Weighted rate factor. As can be seen in Table 14.3, the rate factor is 1.5692 so that the Time Weighted interest for the period of investment is 56.92%.
- Notice the important point that the *time* of cashflow is ignored in the computation. One aspect of any approximation is what is ignored.

An important distinction between the time-weighted and dollar weighted method is that

- The dollar weighted method produces an interest rate, i_D
- The time weighted method produces an interest rate factor, $1+i_T$

14.6: Pricing Stock: What is a stock? A stock is a statement of co-ownership in a corporation. Stock typically arises in situations where initial capital must be raised to start a business. One way of raising money is to float bonds, which are statements of indebtedness. The bond issuer hopes to make enough money to pay back all bond owners and to make profit. A second way of raising money is to sell shares of ownership in the business. For example, if you start a business by issuing 1,000,000 shares of a 10 stock you can raise 10,000,000.

The stock holders or share owners are co-owners. If someone owns a significant amount of stock, that person has voting rights on issues related to the new business. Stocks may be purchased or sold on *exchanges*. Some typical exchanges are the New York Stock exchange the Over The Counter exchange etc. Typically, a person will purchase stock through a broker. The brokerage firm has a *seat* on the exchange. The broker goes to the *floor* of the exchange and mentions the offer (buy or sell) and the price the client wishes. Typically for every desire to buy or sell there is a corresponding desire by someone else to sell or buy respectively. Stocks can also be purchased without a broker say through an online account.

There are many forces dictating the *value* or *price* of a stock. One obvious factor is what people think the stock should be worth. Another approach is to price the stock based on what it gives the owner. Since the stock owner is a co-owner in the corporation, the stock owner may receive *dividends* which are simply earnings of the company. Suppose for example a business worth 10,000,000 makes 2,000,000 during the year. Suppose, as we discussed above, that the 10,000,000 represents 1,000,000 shares of 10. Thus, *per share,* 2 has been made (2,000,000 profit / 1,000,000 shares). Most of the 2 is reinvested in the business to build it up. But the board of directors may decide to say give 0.60 dividend *per share payable quarterly* to all shareholders. That means, that at the end of each quarter, each shareholder receives 0.15 x number of shares. We then have the fundamental pricing method which may be summarized with the following equation.

(14.7) *Price of stock* = Present value *of all future dividends*

Equation (14.7) is known as the *dividend discount model* for stocks.

<u>Illustrative Example</u>: *A stock pays 0.15 every quarter (into perpetuity). Current interest rates are 1.75% annual effective. Price the stock using the dividend discount model.*

<u>Illustrative Solution</u>: $1.0043 = 1.0175^{0.25}$, that is, the rate per quarter is 0.43%. Thus, the price of the stock according to the dividend discount model should be

$$P = 0.15 a_{\overline{\infty}|0.43\%} = \frac{0.15}{0.0043} = 34.51.$$

Typical problems couple the dividend discount model with i) inflation, ii) perpetuities or finite duration annuities, iii) different interest rates in different periods, iv) conversions. The problems are illustrative of these differences.

Illustrative Inflation Example: *A stock will pay a dividend 0.15 at the end of this quarter. In following quarters, the dividend will be 0.1% higher than the previous quarter. The current effective yield is 1.75%. Price the stock using the Dividend Discount Model.*

Illustrative Solution: The timeline is presented in Table 14.8. Equation (14.9) gives the modified rate and the EOV for the price.

Time	0	1	2	3	4	5	…	…
Cashflow		0.15	0.15(1.001)	$0.15(1.001)^2$	$0.15(1.001)^3$	$0.15(1.001)^4$	…	…

Table 14.8: *Cashflows of dividends increasing 0.1% per quarter.*

(14.9) $$\frac{1}{1+i'} = \frac{1.001}{1.0043} = \frac{1}{1.0033} \qquad \text{Price} = v_{0.43\%} 0.15\ddot{a}_{\overline{\infty}|0.33\%} = 44.82$$

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#125 – Dividend discount model

- - - - - - - - - - - -- - - - - - - - - - - - - -

QIT#5 – Dollar Weighted (Same as M03-17)
QIT#78 – Time Weighted
QIT#83 – Time Weighted – Too Simple

QIT#19 – Time and Dollar Weighted (Same as N01-20)
QIT#45 – Time and Dollar Weighted (Same as M01-31)
QIT#120 – Time and Dollar Weighted

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M05-16 Dollar weighted

N00-27 Time weighted 1/2 year vs. whole year

M00-16 time weighted 1st year; dollar 2nd

M01-31 time/dollar weighting one period

N01-20 time/dollar weighting one period

M03-17 Too much computation Bad Problem

CHAPTER 15

Duration

15.1 Overview: You know how to price a bond. You also know how to price a bond based on its book value at a future date. Using a BA II calculator, exact values can be obtained quickly.

However there are several circumstances in which you may wish to quickly approximate changes in prices due to small changes in interest rates. This can be done using a linear approximation. The linear approximations used are similar to the ones you did in Calculus. Approximations are useful for a variety of reasons including i) generally, you should know how changes in variables effect functional values, ii) answering on the spot questions of changes in values, and iii) during simulations when you create several thousand hypothetical interest scenarios.

15.2 A Brief Calculus Review: Let us review how to approximate 1 / 2.01. This was done in Calculus. Clearly ½ is an approximation. The function involved is $f(x) = 1/x$. We know the value of $f(2)=0.5$. Suppose we change $x=2$ by a little bit, say $\Delta x = 0.01$. Calculus encourages one to use the tangent line to the curve $y=f(x)$ at the point $x=2$ to approximate the exact value. The tangent line passes through the point (x=2, y=0.5) and has slope $f'(x) = -1/x^2 = -0.25$. The slope is simply $\frac{\Delta y}{\Delta x} = \frac{\text{Change in } y \text{ value}}{\text{Change in } x \text{ value}}$. This tells us that

$$\frac{-1}{4} = \frac{\Delta y}{\Delta x} = \frac{\Delta y}{0.01} \longrightarrow \Delta y = \Delta x \times \frac{-1}{4} \longrightarrow y(2.01) - y(2) = \Delta y \approx -0.0025 \longrightarrow y(2.01) \approx y(2) - .0025 = 0.4975.$$

The general formula just used is the 1st order Taylor approximation.

(15.1) $$f(x+\Delta x) \approx f(x) + \Delta x f'(x), \qquad f'(x) = \frac{df}{dx}$$

15.3 Approximating the Price Function: We apply (15.1) to a general *portfolio* of cashflows such as presented in Table 15.2. A positive cashflow corresponds to a *deposit* an *inflow* or an *asset;* a negative cashflow corresponds to a *withdrawal* an *outflow* or a *liability.* The price or present value function is presented in (15.3).

Time	0	t_1	t_2	…t_n
Cashflow	0	C_1	C_2	…C_n

Table 15.2: *Cashflows in an investment account*.

(15.3) $$P(i) = C_1 v_i^{t_1} + C_2 v_i^{t_2} + \ldots + C_n v_i^{t_n} = C_1 e^{-t_1\delta} + C_2 e^{-t_2\delta} + \ldots + C_n e^{-t_n\delta}, \qquad \delta = \log(1+i).$$

Price is a function of yield rate, i. Each different yield rate, i, gives rise to a different price. Furthermore, we can easily differentiate $P(i)$ and obtain

(15.4) $$\frac{dP}{d\delta} = -t_1C_1v_i^{t_1} - t_2C_2v_i^{t_2} + \ldots - t_nC_nv_i^{t_n}; \qquad \frac{dP}{di} = \frac{dP}{d\delta}\frac{d\delta}{di} = \frac{1}{1+i}\frac{dP}{d\delta}.$$

We can now apply (15.1) using (15.4). Let us look at the details. First, we let $f(x) = P(i)$. Plugging into (15.1) and then substituting (15.4) we obtain

(**15.5**) $$P(i + \Delta i) \approx P(i) + \Delta i P'(i) = P(i) - \Delta i \frac{1}{1+i}\sum_{j=1}^{n} t_jC_jv_i^{t_j}.$$

15.4 Terminology: Before proceeding further, we introduce some terminology which is summarized in Table 15.6.

Mathematical Concept	$P(i)$	$P(i+\Delta i)$	$P(i+\Delta i) - P(i) = \Delta P$	$\frac{\Delta P}{P}$
English description	Original Price	(Exact) New Price	(Exact) Price Difference	(Exact) Relative price difference

Table 15.6: *Some terminology connected with changes in interest rates*.

The following example illustrates all terms.

Illustrative Example: Suppose a 0-coupon, 1000 dollar, 10-year bond is priced at 5% at time 0. At time t=3, interest rates have increased to 5.1%. Calculate the original price, the exact new price, the exact price difference and the exact relative price difference.

Solution: At t=0, the original price is 1000 x 1.05^{-10} = 613.91. At t=3, the original price (based on the original interest rate) would be the book value, 1000 x 1.05^{-7} = 710.68. Recall the book value, BV_3, is what you could expect to get if you sold the bond at t=3. The *exact new price* is 1000 x 1.051^{-7} = 705.96. *The exact difference in price* is 705.96 – 710.68 = -4.72. The *exact relative price difference* is -4.72/710.68 =-0.66%. Intuitively, the exact relative price difference reflects the *relative* difference. A $4.72 loss on $710.68 is quite different than a $4.72 loss on say a $10 investment.

One more piece of terminology should be introduced. The problem mentions that interest rates increased from 5% to 5.1%. Could we say that interest rates increased 0.1%? Well, we could say that, but someone could misinterpret that to mean that the new rate is 1.001 x 5% = 5.005%. What we really want to say is that there was an *additive* increase in the interest rate. To do this without ambiguity we introduce the concept of *basis points*. 100 basis points is defined as 1%. So the statement that a 5% rate increased 10 basis points unambiguously means that the new rate is 5.1%.

15.5 Refinements of the Basic Approximation: It is customary to subtract P from both sides of (15.5) and divide by P. This gives us a formula for approximating the *relative price difference*. Transforming (15.5) as indicated we obtain

$$(\mathbf{15}.7) \qquad \frac{\Delta P}{P} = \frac{P(i+\Delta i) - P(i)}{P(i)} \approx -\Delta i D(i,1), \qquad D(i,1) = \frac{1}{1+i} \times \frac{\sum_{j=1}^{n} t_j C_j v_i^{t_j}}{P(i)} = -\frac{P'(i)}{P(i)} = \text{Modified Duration}$$

D(i,1) is called the *modified duration;* (1+i) D(i, 1) is called the *Macaulay Duration.* The main benefits of the two durations are as follows:

- *Macaulay duration* is easy to *compute*
- *Modified duration* is what is *needed* to apply the Taylor approximation.

So in practice, to obtain the Modified duration that you need, you first compute the Macaulay duration.

15.7 Duration Computations: To calculate the Macaulay Duration we use (15.4). Let us look at some simple examples.

- Duration of an *n*-year zero-coupon bond: In this case there is only one cashflow *C* at time *n*. Using (15.4) we see that we must compute the price and the present value of this cashflow multiplied by *n*. We have

$$\text{Duration of } n\text{-year 0-coupon bond} = \frac{n \times \text{Present value of } C}{\text{Price}} = \frac{nCv^n}{Cv^n} = n$$

- Duration of a coupon bond: By (15.4), we must compute *t* x PV of each coupon. The computations are laid out in Table 15.7.

Time	1	2	3	…*t*…	*n*	*n*
Numerator of (15.4)	1 x $Fr\,v$	2 x $Fr\,v^2$	3 x $Fr\,v^3$	…*t* x $Fr\,v^t$…	*n* x $Fr\,v^n$	*n* x Cv^n

Table 15.7: *Computation of summands in numerator of (15.4) for a coupon bond.*

We recognize the bottom row as the present values of an increasing annuity. So we have

$$\text{Duration of coupon bond} = \frac{Cnv_i^n + \sum_{j=1}^{n} jFrv_i^j}{\text{Price}} = \frac{Cnv_i^n + Fr(Ia)_{\overline{n}|i}}{\text{Price}}.$$

Many more formulas may be derived and some should be of interest to those who take the SOA exam.

15.8 Illustrative Examples: Let us return to the illustrative example presented in Section 15.4. Corresponding to the *exact* entities we can now compute *approximate entities.* More specifically, we can compute the *approximate relative price change, approximate price difference,* and *approximate new price.*

- *Approximate Relative Price Change:* Using (15.6) we obtain -0.1% x 7/1.05 = -0.67%. The exact relative price change is -0.66% so the approximation is quite good.
- *Approximate price difference:* We take *approximate relative price change* x *exact original price*. We obtain -0.67% x 710.68 = -4.74. This compares well with the *exact price difference* of -4.72.
- *Approximate new price:* We simply add the approximate price difference to the original price. We obtain 710.68-4.74=705.94 which compares well with the *exact new price* of 705.96.

15.9 Project: Accompanying this chapter is a Chapter 15 spreadsheet. Each student is coded with the first letter of their first name and the last two letters of their last name. So my code would be RHe. My version of the project as given in this chapter would start with the five values *n*=10, *C*=1000, reprice time =3, *i*=5%, interest rate change=10 basis points. Using these five values we can compute the price of the bond at 0 and its expected reprice at 3 based on the original interest rate. We can also compute the *exact new price, exact price difference* and *exact relative price difference*. Then using the Macaulay duration of an *n*-year bond we can compute the *approximate relative price difference*, *approximate price difference* and *approximate new price*. One can also compare the exact and approximate values to double check on accuracy.

Each student is to take *their* values and perform the project. The get credit the steps should be numbered in the following order with final answers underlined or boxed in: 1) Exact price at 0, 2) Exact reprice at reprice time, 3) exact new price, 4) exact price difference, 5) exact relative price difference, 6) Macaulay duration, 7) Modified duration, 8) Approximate relative price difference, 9) Approximate price difference, 10) Approximate new price. You should check your own work and make corrections if your approximations are a bit off. *To get full credit you must show a) algebraic equations, b) numerical equations, and c) final numerical answer for all parts* *except part 6* (In other words, since the Macaulay duration is n it suffices to write 6) n = #. No further work need be shown).

15.10 Miscellaneous: This chapter has covered a lot of ground for just one concept. However, there are many more concepts. We therefore just touch on five of them.

- Formulas for duration: We have already mentioned in Section 15.7 that there are many exact duration formulas for specific situations such as buying a bond for its redemption value. My feeling is that these formulas would be of interest to those who take SOA exams but are not relevant to the main theory.

- The Portfolio Approach: If one carefully looks at the formula for Macaulay duration (15.7) one sees that it is a weighted average of present values of cashflows. One implication of this is that one can compute the duration of a portfolio by taking a weighted average of its components without using (15.7). For example, if portfolio A has a present value of 100 and a duration of 3 while portfolio B has a present value of 200 and a duration of 6 then the entire portfolio has a present value of 300 with A and B representing 1/3 and 2/3 of the entire portfolio present value.
  Entire combined portfolio of A & B duration = 1/3 x 3 + 2/3 x 6 = 5.
  The duration can be computed without using (15.7) directly. See M05#6 for another example.
- Convexity: We used (15.1) to derive (15.5) and (15.7). But (15.1) is a *first order* Taylor series. We could theoretically have used a second order Taylor series. We obtain
  $$P(i+\Delta i) = P(i) + \Delta i P'(i) + \frac{1}{2}P''(i)\left(\Delta i\right)^2; \qquad P''(i) = \frac{d^2P}{di^2}$$
- After subtracting $P(i)$ and dividing by $P(i)$ we obtain
  $$\frac{\Delta P}{P} = \Delta i\frac{P'(i)}{P(i)} + \frac{1}{2}\left(\Delta i\right)^2\frac{P''(i)}{P(i)}$$
  This motivates a concept of *modified convexity* defined by
  $$\frac{P''(i)}{P(i)} = \text{Modified Convexity}$$
  As before there is also a concept of Macaulay Convexity which we will define in a minute. The idea is that Modified Convexity is what is needed for Taylor series approximations while Macaulay Convexity is what is needed for easy computation. In analogy with (15.4) we have
  $$\textbf{(15.8)} \qquad \text{Macaulay convexity} = \frac{t_1^2C_1v_i^{t_1} + t_2^2C_2v_i^{t_2} + \ldots + t_n^2C_nv_i^{t_n}}{\text{Price} = P(i)}$$
  So Macaulay Convexity uses multiplication by t^2 while Macaulay Duration uses multiplication by t
- Semi-annnual payments: The exposition above is done for annual payments. Those studying for the SOA exam may wish to learn the other types of duration. I feel that for a small topic the above exposition is sufficient.

- Equation (15.4) vs (15.5): If one looks closely at (15.7) one sees two definitions of duration; one based on a weighted average of present values and one based on derivatives (-P'(i)/P(i)). Certain problems lend themselves to the derivative definition. This will be illustrated in the exercises such as QIT#36 and QIT#37.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#35 – Duration
QIT#36 – Duration stock – Derivative method
QIT#37 – Duration stock – Derivative method
QIT#59 – Duration
QIT#65 – Duration

QIT#66 – Duration – Change in Price
QIT#68 – Duration
QIT#121- Duration (Algebra)
QIT#122 – Macaulay Duration – Plugin
QIT#123 – Macaulay Duration – Bond bought at par
QIT#124 – Duration stock – Derivative Method

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M05#3 Duration - Bond

N05#2 Duration - Bond

M05#6 Portfolio Duration

CHAPTER 16

Exact Asset Matching, Immunization

16.1 Review: Prior to presenting the main topic for today we review the idea of a *portfolio* and the associated *price function.* These were presented in Chapter 15, Table 15.2 and EOV (15.1).

A general *portfolio* of cashflows is presented in Table 16.2. A positive cashflow corresponds to a *deposit* an *inflow* or an *asset;* a negative cashflow corresponds to a *withdrawal* an *outflow* or a *liability*. More precisely,

(16.1) $C_j > 0 \longrightarrow C_j$ is an asset; $C_j < 0 \longrightarrow |C_j|$ is a liability.

The general timeline of a portfolio is as follows.

Time	0	t_1	t_2	$\ldots t_n$
Cashflow	0	C_1	C_2	$\ldots C_n$

Table 16.2: *Cashflows in an investment account.*

The EOV corresponding to Table 16.2 is presented in (16.3) which presents three important functions: the *price, asset* and *liability* function. Each function is a function of interest rate, *i*.

(16.3) $P(i) = C_1 v_i^{t_1} + C_2 v_i^{t_2} + \ldots + C_n v_i^{t_n} = A(i) - L(i);\quad A(i) = \sum_{C_j>0} C_j v_i^{t_j};\quad L(i) = \sum_{C_j<0} |C_j| v_i^{t_j}$

Each distinct interest rate *i,* gives rise to a different price. This allows us to *graph price* as a function of *interest* and apply the calculus theory of extrema. Let us first briefly review calculus' approach to extrema. Figure 16.4 contains a typical graph. Note the local minima at $x=i_0$. Calculus explains how to recognize such a minima. The explanation is summarized in Table 16.5

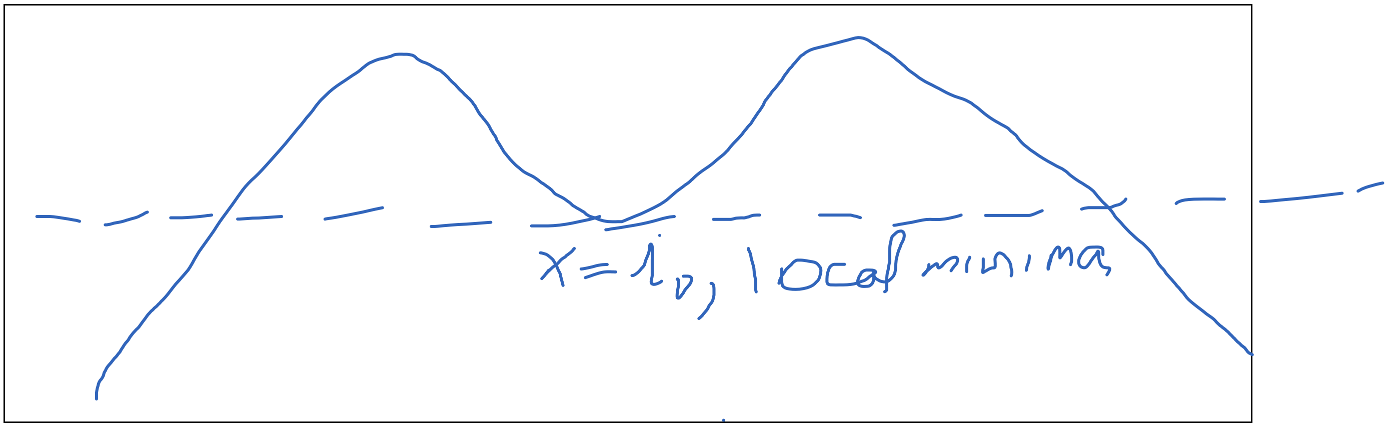


Figure 16.4: *Graph with local minima at $x=i_0$. The criteria for recognizing a local minima may be found in Table 16.5*

Requirement for minima	**Mathematical description**
Flat tangent line	$f'(i_0) = 0$
Curve is concave upward, *holding water*	$f''(i_0) > 0$
Local minima	Tangent line to minima does intersect curve in other places
Global minima (Think parabola, like letter U)	Tangent line to minima does not intersect curve elsewhere

Table 16.5: Requirements for minima: English and mathematical descriptions.

16.2 Overview of Asset Matching: If you run a business you frequently encounter large liabilities. For example, if you use equipment – computers, manufacturing equipment, shipping equipment, etc. – your present computer or equipment may only be expected to last so many years. This means, that say, five years from now you will have to purchase new equipment. You wouldn't want to suddenly have to pay several million dollars that year and nothing the years prior; it would create a distorted sense of the company's financial position. Furthermore, you might have other distinct liabilities in distinct years.

The payments that must be made for the new machinery are called *liabilities*. *Liabilities* can arise in other manners (for example, litigation). To prepare for the liabilities, the company would purchase *assets*. There are three approaches we will study for purchasing and matching assets to liabilities. They are summarized in Table 16.6.

Approach name	**Requirement #1**	**Requirement #2**	**Requirement #3**	**Requirement #4**
Exact asset matching	Assume *one* interest rate holds and remains for several years	$A(i) = L(i)$ (Hence name *exact*)		
Reddington (local) immunization (P(i) will not go down for *small* changes in interest rates)	Allow assumption of changing interest rates	$P(i)=0$; alternatively $A(i)=L(i)$	$P'(i)=0$; $A'(i)=L'(i)$, alternatively Duration(A) =Duration(L)	$P''(i) > 0$; $A''(i) > L''(i)$, alternatively, Convexity(A) > Convexity (L)
Full (Global) immunization (P(i) cannot go down; you are fully immunized against loss)	Allow assumption of changing interest rates	$P(i)=0$; alternatively $A(i)=L(i)$	$P'(i)=0$; $A'(i)=L'(i)$, alternatively Duration(A) =Duration(L)	Two assets for every liability, one maturing prior to liability date and one maturing after liability date

Table 16.6: *Three approaches covered in this chapter to matching expected liabilities with assets*

To fully understand Table 16.6 we should explain the *alternative* formulations. First using (16.3) we have

$P(i) = A(i) - L(i)$ and $P'(i) = 0 \longrightarrow 0 = P'(i) = A'(i) - L'(i) \longrightarrow A'(i) = L'(i)$.

Next using i) the definition of duration, ii) the assumption $A(i)=L(i)$, and iii) $P'(i)=0$ we have

$$P'(i) = 0 \longleftrightarrow A'(i) = L'(i) \longleftrightarrow \frac{A'(i)}{A(i)} = \frac{L'(i)}{L(i)} \longleftrightarrow \text{Duration}(A) = \frac{A'(i)}{A(i)} = \frac{L'(i)}{L(i)} = \text{Duration}(L)$$

16.3 Exact Asset Matching: I have selected **QIT#69** to illustrate the method since a table template is used by the SOA in stating the problem. We have modified the problem to fully illustrate the method.

Illustrative problem (**QIT#69**): An insurance company must pay liabilities of 99 at the end of one year, 102 at the end of two years and 100 at the end of three years. The only investments available to the company are the

following three bonds with par value of 100 redeemable at par. Bond A and Bond C are annual coupon bonds with 7% and 5% coupons. Bond B is a zero-coupon bond. Calculate the number of A, B, and C bonds needed to exactly match assets. (Details about the bonds may be found in Table 16.7)

Illustrative Solution: The following *asset-liability* Table, 16.7, presents what is given and illustrates what must be done in setting up the problem. In creating this table we create the following variables:

- *A* represents the number of A bonds bought
- *B* represents the number of B bonds bought
- *C* represents the number of C bonds bought

Variable meaning and definition is a crucial and important part of any problem as we will discuss below.

To clarify the construction of the table we interpret the C row. Suppose you buy *C*, C-Bonds. Each single C bond gives a coupon of Fr=5% x 100 = 5. So if you buy *C* such bonds you receive at times t=1,2, 5*C*. At time *t*=3 besides the 5*C* you also receive the 100 redemption value for each C bond bought.

The *exactness* of *asset-liability matching* requires for each point in time that the liabilities for that point of time exactly equal the assets. Hence *each column generates an equation* as shown.

Time	0	1	2	3

Liability due at *t*	0	99	102	100
Assets		=	=	=
A Maturity=1, *i*=6%, *r*=7%		107*A*		
B Maturity =2, *i*=7%			100*B*	
C Maturity =3, *i*=9%, r=5%		5*C*	5*C*	105*C*
Equations (Read each column vertically)		99=107*A*+5*C*	102=100*B*+5*C*	100=105*C*

Table 16.7: *Asset-Liability Table template for illustrative problem.*

How do you solve three equations in three unknowns. One very powerful method is *backtracking.* You first solve the equation (if any) that has one variable. Then you plug in and solve any equations with one more variable. You continue this process till all equations are solved.

We see that Equation 1 (of column 1) has 2 variables. But equation 3 has 1 variable. Hence, we do as follows

- I can solve Equation 3, 100=105*C*,→ *C* = 100/105= 0.9524
- Using this value of C, Equation 2 becomes 102=100*B*+5*C*=100*B*+5(0.9524) → *B*=0.9724
- Using these two values, Equation 1 becomes 99=107*A*+5*C* = 107*A*+5(0.9524) → *A*=0.8807.

Thus, the solution to the illustrative problem is to buy 0.8807 units of A, 0.9724 units of B and 0.9524 units of C.

To fully illustrate this method we present a second illustrative problem.

Second Illustrative Problem: With assumptions as in the illustrative problem, calculate how much must be spent on A,B, and C bonds to exactly match assets.

Note: Note the following subtlety. The original illustrative problem asked for the *number* of A,B, and C bonds while the second illustrative problem asked for the *cost* of the A,B, and C bonds. To solve the second illustrative problem, it might seem logical to let *A*,*B*, and *C* be variables, which represented *the number of bonds* bought for the illustrative problem, representing *cost*, that is, PV, or *price*. Unfortunately, this is not true. It is much easier to solve for *number* than *cost.* If a problem asks for *cost* you should *not answer the question directly.* Instead calculate the *number* needed and then calculate the *cost*, PV separately.

Illustrative Solution: As just indicated we begin the solution of the 2nd illustrative problem identically to the solution of the original illustrative problem. However, *after* solving for the number of bonds required to be bought, we must *additionally* solve for price.

We illustrate the technique for solving for price for bond C, the other two cases – A and B – being similar and easier. Calculating the price of a bond is a straightforward matter that we have done several times. But because we are in another module we are liable to forget the technique. 1st a timeline of cashflows,Table 16.8, is made:

Time	0	1	2	3
Cashflow	-*P* (total cost)	5*C*=5(0.9524)=4.76	4.76	(0.9524)*(100+5) = 100

Table 16.8: *Timeline of cashflows for Bond C.*

The price can be calculated by the basic bond EOV, (16.9), and the traditional TV line, presented in Table 16.10.

We conclude that to exactly match liabilities, 92.54 must be spent on bond C.

$$(\mathbf{16.9}) \qquad P = 0.9524\left(5a_{\overline{3}|9\%} + 100v_{9\%}^{3}\right)$$

N	I	PV	PMT	FV
3	9	CPT=85.59	4.76 =(0.9524)*5	95.24 =(0.9524)*100

Table 16.10: *TV line to calculate the total cost of Bond 0.9524 units of Bond C.*

16.4 Local (Reddington) Immunization: Both the assumptions and requirements of local immunization as well as what it accomplishes are summarized in Table 16.6. The example given in Section 16.5 illustrating full (global) immunization will also be used to illustrate local (Reddington) immunization. Hence, we proceed directly to Section 16.5 and full immunization. Furthermore, Example 1 in Section 16.7 illustrates how to approach a local immunization problem using the global immunization techniques.

16.5 Full (Global) Immunization: To illustrate the basic technique we use the following illustrative example.

Illustrative Example. A firm has a liability of 100,000 in 7 years. The market offers 0-coupon bonds maturing in 3 and 10 years at the current market rate of 2%. How much of the 3-year and 10-year bonds must be purchased now to fully immunize against the liability of 100,000 at time 7.

Illustrative Solution: Before applying the method, we must assure that Requirement 4, from Table 16.6 is satisfied. Indeed it is, since there are two assets for every liability with one asset maturing at 3 years, prior to the 7 year liability date, and one asset maturing at 10 years, after the 7 year liability date.

We now proceed to the solution. I am indebted to Daniel-Vaaler for this easy 6 step solution. It turns out the Daniel-Vaaler approach should be used on any immunization problem. Using the Daniel-Vaaler method first will assure an easy solution to any problem. The 6 step solution is as follows:

Step 1 : Let x = the total price of the 3 year bonds, Let y = the total price of the 10 year bonds

Step 2 : The requirement $A = L$ from Table 16.6 means $x + y = 100000v_{2\%}^{7} = 87056.02$

Step 3 : The requirement Duration(A)=Duration(L) from Table 16.6 means Duration(Two Assets) = Duration(L)

Step 4 : Duration(L) = 7, $\text{Duration}(A) = 3\frac{x}{x+y} + 10\frac{y}{x+y}$

Step 5 : By Steps 3 and 4, $7 = 3\frac{x}{x+y} + 10\frac{y}{x+y} \longrightarrow 7(x+y) = 3x + 10y \longrightarrow 4x = 3y \longrightarrow x = \frac{3}{4}y$

Step 6 : By applying Step 5 to Step 2, $87056.02 = x + y = 1.75y \longrightarrow y = 49746.30 \longrightarrow x = 37,309.72$

Recall from Table 16.6, that local immunization requires checking that *P*"(i)>0 while global immunization does not so require. Let us take our example: i) A 100,000 liability maturing in 7 years, ii) An asset of 37,309.72 in three year 0-coupon bonds, iii) an asset of 49746.30 in 10 year bonds, and iv) a current interest rate of 2% and *verify* that the requirements of local immunization are present. This is done in Table 16.11. In deriving the equations in Table 16.11 recall that i) (Macaulay) Convexity is a weighted average of *squares* of times while (Macaulay) Duration is a weighted average of time, ii) Duration of a portfolio is the weighted average of duration of portfolio components, the weights being the percentage of price of the PV of that asset, iii) (Macaulay) duration of an *n*-year 0-coupon bond is *n*.

Requirement for Reddington Immunization (Table 16.6)	Verification of Requirement
$P(i)=0$ or $A(i) = L(i)$	49746.30+37309.72 = 87056.02 = PV(L of 100,000)
Duration(A) = Duration(L)	7= 37309.72/87056.2 x 3 + 49746.30/87056.2 x 10
Convexity(A) > Convexity(L)	$49=7^2<37309.72/87056.2 \times 3^2 + 49746.30/87056.02 \times 10^2 = 61$

Table 16.11: *Using this section's example to illustrate the requirements of Reddington Immunization.*

To recap: Whether you are given a local or global immunization problem you *begin* the solution using the requirements A=L and Duration(A)=Duration(L). You solve this problem using the six step Daniel-Vaaler method. Then, if Requirement 4 in Table 16.7 for global immunization is present you are done. If not, then you must check Requirement 4 in Table 16.7 for local immunization. Example 1 in Section 16.7 illustrates this fundamental technique.

16.6 Immunization Project: Accompanying this chapter is a spreadsheet with personal immunization projects for each student. Again, each student is coded with three letters, their first initial and the first two initials of their last name. Next to each student code are five numbers: i) The liability (e.g. 100000), ii) when it is due (e.g. 7), iii) The maturity date of the first zero coupon bond available for purchase (e.g. 3), (iv) the maturity of the 2nd zero coupon bond available for purchase (e.g. 10), v) the current interest rate (e.g. 2%).

To complete the project each student should do the following:

- (Step 1a) State their goal of full immunization by setting *A*=*L* and *A'*=*L'* and verifying that (Step 1b) one maturity date falls prior to the liability due date while the other maturity date falls after the liability date
- (Step 2) Review the derivation that *A'*=*L'* can be satisfied by Duration(*A*)=Duration(*L*) provided *A*=*L*.
- (Steps 3 – Steps 8) Go through the six steps of the derivation above
- (Step 9) Finally check that the convexity of the assets is greater than the convexity of the liabilities.

Please label all the steps and do the work linearly.

16.7 Challenging Immunization Problems: I indicated above that the Daniel-Vaaler six-step method is all you will ever need in practice and it makes all problems very simple. Here are some very difficult problems that can be solved by this method.

Example 1: Immunize liabilities of 10000 at t=1 and 20000 at t=2, using zero-coupon bonds maturing at t=1.5 and t=4. The current interest rate is 4%.

Solution and Discussion: This problem is not a full immunization problem since there is no bond maturing prior to the first liability (Requirement 4 of Table 16.6 for full immunization). Therefore local immunization methods must be used. The problem therefore appears difficult. It used to be classified as ADAPT Level 7.

But if we *approach* this problem using full-immunization *methods* the problem becomes easy. After all, to solve local immunization, according to Table 16.6, one needs to verify 3 requirements, 2 of which coincide with the full immunization methods. Furthermore we are still free to:

- Define variables by the PV amounts (price) rather than redemption value
- Use equality of duration instead of equality of derivatives
- Write the denominator of price as the sum of PVs (thereby facilitating solving for one variable in terms of the other)
- Check for convexity relations after solving the equations.

Here are the details (Carefully compare the six steps here to the six steps above)

Step 1 :	Let x = the total price of the 1.5 year bonds,	Let y = the total price of the 4 year bonds
Step 2 :	The requirement $A = L$ from Table 16.6 means	$x + y = 10000v_{4\%} + 20000v_{4\%}^2 = 28,106.51$
Step 3 :	The requirement Duration(A)=Duration(L) from Table 16.6 means	Duration(Two Assets) = Duration Two Liabilities
Step 4 :	Duration Liabilities $= 1 \times \frac{10000v}{28106.51} + 2 \times \frac{20000v^2}{28106.51} = 1.657895,$	$\text{Duration}(A) = 1.5\frac{x}{x+y} + 4\frac{y}{x+y}$
Step 5 :	By Step 4, $1.657895 = 1.5\frac{x}{x+y} + 4\frac{y}{x+y} \longrightarrow$	$1.657895(x+y) = 1.5x + 4y \longrightarrow 0.157895x = 2.342105y \longrightarrow x = \frac{2.342105}{0.157895}y = 14.8333y$
Step 6 :	By Applying Step 5 to Step 2, $28106.51 = x + y = 15.83333y \longrightarrow$	$y = 1775.15 \longrightarrow x = 26331.36$

To complete the problem we must check Requirement 4 of Table 16.6 which asserts Convexity(A)>Convexity(L)

$$\text{Convexity}(L) = 1^2 \times \frac{10000v}{28106.51} + 2^2 \times \frac{20000v^2}{28106.51} = 2.9737, \qquad \text{Convexity}(A) = 1.5^2\frac{26331.36}{28106.51} + 4^2\frac{1775.15}{28106.51} = 15.1316$$

Clearly Convexity(A) = 15.1316 > 2.9737 = Convexity(L), as required for local immunization.

Example 2: A 50,000 liability is due at time *x*. To fully immunize, there are two assets available for purchase: One asset matures at time *x-t* for A while the other asset matures at time *x+t* for B. Calculate the values of A and B in terms of *t*.

Solution & Discussion: Notice that the problem is naturally formulated in terms of redemption values. Such a formulation leads to a problem that was classified by ADAPT at Level 9. However, this problem is easy if the Daniel-Vaaler approach is used formulating in terms of PV. We solve for the PV and *then* solve for A, B. We simply sketch one part of the solution remembering that the hard part is *how* you formulate it.

The requirement from Table 16.6 that *A*=*L*, when formulated using PV becomes

$$a + b = 50000e^{-\delta x}$$

The requirement from Table 16.6 that Duration(*A*)=Duration(*L*) when formulated using PV becomes

$$\frac{a}{a+b}(x-t) + \frac{b}{a+b}(x+t) = x$$

We then solve (by clearing denominators, solving for one variable and plugging into the *A*=*L* equation to solve for the other variable) to obtain

$$2a = 50000e^{-\delta x}, \qquad a = Ae^{-\delta(x-t)} \longrightarrow A = 25000e^{-\delta t}$$

Here little *a* and *b* correspond to the present values of the redemption values of capital A and B. The problem was easy to solve because we used the Daniel-Vaaler method which includes

- Letting variables *x* and *y* represent PV
- Replacing equality of derivatives with equality of duration
- Formulating denominators as *x+y* (instead of *P*) facilitating solving for one variable in terms of the other
- Solving for one variable in terms of the other and then using the *A*=*L* equation to solve for that variable.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#51 Asset Matching

QIT#52 Asset Matching

QIT#53 Asset Matching

QIT#69 Asset Matching

QIT#132 Asset Matching – Tree – Too difficult

QIT#133 Asset Matching – Tree – Too Difficult

QIT#59 Immunization

QIT#70 Immunization

QIT#71 Immunization

QIT#72 Immunization

QIT#73 Immunization

QIT#127 Immunization

QIT#128 Immunization

QIT#129 Immunization

QIT#130 Immunization – Annuity
QIT#131 Immunization – Decision Tree

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

M05#15 Asset Matching

N05#10 Asset Matching

N05#21 Immunization

CHAPTER 18

Spot Rates, Forward Rates, Yield Curve

18.1 Prime and LIBOR Interest Rates: Chapter 20 will discuss determinants of interest rates. But for purposes of this chapter we need to know some basics.

Suppose Bank of America needs $10,000,000 to finish some business. It might go to Chase and *loan* the 10,000,000 with the intent of paying it back the next day or so. Chase will loan the money. It will charge interest on the loan. Interest is sometimes called the *rental cost* of money since Chase is in effect renting the value of the money to Bank of America.

The important point to emphasize is that the interest rate that Chase charges Bank of America is much less than the rate it might charge some small bank or charge you or me for a loan to buy a car or house. Why? It is very unlikely that Bank of America will *default* on its obligation to pay loans. If Chase does not expect defaults on loans it can charge less interest since if it does expect defaults it will have to charge more to make up for losses on default loans. We will learn about this interesting perspective in Chapter 20.

It follows from this discussion that the rates that big banks charge each other are important rates and are typically less than other rates. We call these rates the *prime rate.* You can actually look up daily in newspapers and the internet the *prime rates.*

In Europe, the average rate that banks charge each other is called the LIBOR, the London Interbank Overnight Rate. It functions like the prime rate. It too is published daily.

18.2 Fixed and Variable Rates: Throughout the term we have discussed *fixed* interest rates. For example, suppose you take out a loan at 5%. That 5% interest rate and its consequent obligations remain the *same, level* or *fixed* throughout the term of the loan.

However, certain loans are *variable.* Some of your credit cards may charge you *variable* rates. A typical variable rate might be the prime rate + X bps where bps stands for basis points (hundredths of a percent which we discussed in Chapter 15). If you loan money on a credit card and that credit card uses a variable rate then your obligation will depend on the prime rate. If the prime rate goes up (or down) your rate (which is X bps higher or lower) will change also.

18.3 Spot Rates: Generally, you can make more money over a longer time period than a shorter time period. Therefore, you should be willing to give a higher rate of return over a longer period than over a shorter period. The fixed annual rate you would charge for a zero-coupon bond of 1 maturing in *n* years is called the *n-year spot rate.* Think of it as the rate you would get *on the spot,* or immediately. If you are a bank or other institution lending money, the *collection* of spot rates for the next *n* years is called the *term structure of rates.* A typical possible term structure is presented in Table 18.1

Term to maturity	1	2	3	4	5
Spot rate, r_n	2%	3%	3.5%	3.75%	3.8%
P_n, Price of 0-coupon *n* year bond maturing for 1. Note that P_n=v(*n*), the discount factor of \$1 from *t*=*n* to *t*=0	0.9804	0.9426	0.9019	0.8631	0.8299

Table 18.1: *Hypothetical term structure of rates for 5 years. Prices of 0-coupon bonds maturing for 1 are added.*

Notice that the *term structure of rates* can be thought of as a collection of number pairs (t, r_t). If we add the point (0,0) and graph this collection of numbers we obtain the *yield curve, spot rate curve,* or *zero-coupon curve.*

We can then speak about the *shape* of the yield curve. Three typical shapes are listed in Table 18.2

Description of curve	Increasing, concave downward	Flat line	Decreasing, concave upward
Name of curve	Normal curve	Flat curve	Inverted yield curve

Table 18.2: *Yield curves classified by shapes*

An important point to remember is that a flat curve means spot rates for all terms of maturity are the same. The reason why it is *normal* for the spot rate curve to increase with a tapering off will be explained in Chapter 20 in our discussion of determinants of interest.

18.4 Spot rates, Zero coupon bond prices, Discount factors: There are three ways to indicate the annual discount factor needed to compute present values. They are presented in Table 18.3

Method of indicating PV	Discount factor $v(2)$	Spot rate, r_n	P_n=Price of *n*-year 0 coupon bond
Example: $i=3\%$, $n=2$	$v(2)=1/(1+i)^2=1/1.03^2=0.9426$	$1/(1+r_n)^2$ = $1/1.03^2=0.9426$	$P_2=1/1.03^2=0.9426$

Table 18.3: *Three methods of giving the PV.*

Some important points should be noted related to Table 18.3. Suppose you are told that the price of a 2-year, 0 coupon bond is 0.9426. Suppose further there is a cashflow of 10 at time t=2. How do you compute the PV of this cashflow.

You might be tempted to solve for i. So you would take the square-root of 0.9426. You would then flip it and subtract 1 to obtain the interest rate. You would then calculate $v = 1/(1+i)$. You would then compute $10v^2$.

But when squaring v you obtain 0.9426, your original number. The point of this exercise is to show you that the price already equals v^2. You don't have to do anything else. The cashflow at 10 has a PV of 10 x 0.9426 = 9.43.

A second important point is that the spot rate is an annual rate. This can be confusing so let us compare the spot rates of 3% and 3.5% for t=2 and t=3 from Table 18.1. The 3% spot rate means that the buyer of the bond gets 3% in both the first year and 2nd year. The 3.5% means that the buyer of the bond gets 3.5% in years 1,2 and 3. It is a mistake to say that the buyer gets 2% in the first year, 3% in the second year and 3.5% in the 3rd year. This mistake is so common that we lay out the consequences for each year in Table 18.4

Term to maturity	**Spot rate, r_n**	**Interest in year 1**	**Interest year 2**	**Interest year 3**
1	2%	2%	*NA*	*NA*
2	3%	3%	3%	*NA*
3	3.5%	3.5%	3.5%	3.5%

Table 18.4: *Annual rates for various zero-coupon bonds.*

18.5 Forward Rates: Consider the term structure exhibited in Table 18.1. Suppose I want to loan someone 5 for two years at t=1. What rate should I plan to charge him now at $t=0$.?

We don't immediately know the answer to this so let us call the rate $f_{1,3}$. Here the f means the rate going *forward* one year. The subscript 1,3 refers to the annual rate from t=1 to t=3. Can we say anything about $f_{1,3}$. We first look at the timelines of Table 18.5

Time	0	1	2	3
Rate factor to next year	$1+r_1$	$1+f_{1,3}$	$1+f_{1,3}$	
Rate factor	$1+r_3$	$1+r_3$	$1+r_3$	

Table 18.5: *Two ways to describe annual rates for the next three years.*

The two descriptions of rates must be equal or else people could exploit the difference to make money without investing. We therefore obtain the following EOV:

$$\textbf{(18.6)}\quad TV_1 = TV_2 \longrightarrow (1+r_1)(1+f_{1,3})^2 = (1+r_3)^3 \longrightarrow \frac{1}{(1+f_{1,3})^2} = \frac{1+r_1}{(1+r_3)^3} = \frac{P_3}{P_1} = \frac{0.9019}{0.9804} = 0.9200$$

But what is $f_{1,3}$? I don't care! I just want the PV factor from t=1 to t=3 and I don't even need it for one year; having a present value factor for two years is all I need. I can now solve my problem. Here is the solution. Note the English.

The price at t=1 computed at t=0 of a 1-year deferred loan of 5 for 2 years at the current term structure is 0.92 x 5 = 4.60

Again we emphasize: The loan is priced based on

- The term structure at time *t*=0
- An initial payment date of *t*=1

Illustrative Problem: Let us use the above concepts to price a special annuity. *Using the term structure in Table 18.1, price a one-year deferred annuity with payments of 10 at times 1 and 3 years after purchase.* The timeline is presented in Table 18.7.

Time	0	1	2	3	4
Cashflow		Price, outflow	10, inflow		10, inflow

Table 18.7: *One-year deferred annuity with payments of 10 at times 1 and 3 years after purchas, priced at 0.*

Illustrative Solution: There are many ways to solve this problem. You, the student, wants a solution that is not confusing and uses intuitive notation. For this, we use the discount factor notation introduced above. Recall, for example, that *v*(2) represents the discount factor for *t*=2, that is, the amount you would be pay at *t*=0, under the current term structure of interest rates, to obtain 1 at *t*=2. We use *v*(2) rather than v^2 because we are not taking powers. And why are we not taking factors. Because when we deal with a term structure we no longer have a level yield and each *v*(*t*) is "separate."

The illustrative solution is "easier" if we discount all present values to *t*=0 (it is more complicated to make *t*=1 the "new 0" as we do when we have level rates). The illustrative solution is presented in Figure 18.8

```
Present Value of outflow  = Present value of inflows

Present value of Price (at t=0) = Present Value of $10 inflows

P v(1) = 10 v(2) + 10 v(4)
```

$$P\ P_1 = 10\ P_2 + 10\ P_4$$

$$9804\ P = 9426 \times 10 + 8631 \times 10 \Rightarrow P = 18.4180$$

Figure 18.8: Elegant and compact solution to the illustrative problem in a step by step format.

The generalization of (18.6) is presented in (18.9).

$$(\mathbf{18.9}) \quad TV_1 = TV_2 \longrightarrow (1+r_u)^u(1+f_{u,w})^{w-u} = (1+r_w)^w \longrightarrow \frac{1}{(1+f_{u,w})^{w-u}} = \frac{(1+r_u)^u}{(1+r_w)^w} = \frac{P_w}{P_u}$$

As a final note, *forward rates* are also called *theoretical forward rates* or *implied forward rates.* The names reflect the fact that one only sees in the market, newspapers and internet spot rates and not the actual forward rates.

Section 18.6 Strips: If you have a bond with coupons and there is non-flat yield curve you can make money off each coupon as follows: You can sell the coupon at time t=1 for the one year spot rate, the coupon at time t=2 for the 2 year spot rate,…,the redemption value at time n for the n year spot rate. Each of these coupons is called a *strip.* Intuitively, you can think of the bond as piece of paper with little coupons that you tear off and redeem. So these coupons are *stripped* off the bond.

A reverse problem is to price a bond from component strips. Here is an illustrative example.

Illustrative Example: *Strip redemption values are as follows:*

- *1 year strip for $50*
- *2 year strip for $50*
- *3 year strip for $50*
- *Redemption value strip at 3 for $1000.*

Regard this collection of payments as a bond, *price this bond at the current term structure using the Term Structure in Table 18.1 and calculate the implied level effective rate.*

Illustrative Solution: The term structure is presented in Table 18.1. Note, as before, we can calculate present values using the prices of zero coupon bonds presented in Table 18.1. We have

- Price of 1 year strip of $50 is 50 $v(1)$ = $50P_1$ = 50 x 0.9804 =49.02
- Price of 2 year strip of $50 is 50 $v(2)=50P_2$=50 x 0.9426 = 47.13
- Price of 3 year strip of $50 is 50 $v(3)=50P_3$ = 50 x 0.9019 = 45.10
- Price of redemption value strip of $1000 is 1000 $v(3)=1000\ P_3$ = 1000 x 0.9019 = 901.90
- Price of bond is sum of prices of all its strips: 49.02+47.13+45.10+901.90 = 1043.15.

Table 18.10 calculates the level effective yield of the bond using a traditional EOV and TV line.

N	I	PV	PMT	FV	Comment
3	CPT 3.4610%	-1043.15	50	1000	Traditional TV bond line

Table 18.10: *TV line to calculate yield of bond in illustrative example*

We have calculated it. But what does it mean. What *is* 3.4610%? The question can be answered by a cashflow diagram. If I deposited 1043.15 in a bank yielding 3.4610% I can meet all obligations corresponding to the strips and exhaust the account. This is illustrated in Table 18.11

Time	0	1	2	3
Beginning account value	0	1043.15	1029.25	1014.88
Cashflow	1043.15	-50	-50	-50 coupon -1000, redemption
Interest		36.10=1043.15 x *i*	35.62=1029.25 x *i*	35.12 =1014.88 x *i*
Net Account value		1029.25	1014.88	0

Table 18.11: *Cashflow showing that 1043.15 at 3.4610% is sufficient to meet all strip obligations.*

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/

QIT Problems:

QIT#33 Bond Spot Rate

QIT#34 Bond Spot Rate

QIT#67 Spot-Forward

QIT#92 Forward Rate
QIT#119 Spot-Forward

FROM THE ARCHIVED EXAMS: No solution posted.

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)

https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

N05#15 Spot Forward Rates

M05#10 Spot Forward Rates

N05#19 Spot Forward Rates

CHAPTER 19

Swaps

19.1 Yield Curve: We borrow Table 18.1 from the last chapter. Table 18.1 is re-presented below in Table 19.1

Term to maturity	1	2	3	4	5
Spot rate, r_n	2%	3%	3.5%	3.75%	3.8%

Table 19.1: *Term structure* or *yield curve: Spot rates for 0-coupon bonds with maturities of 1-5 years.*

You will find it very convenient to add rows for prices and one year forward rates. A bit about notation. The one year forward rate for year *i*, is $f_{i-1,i}$. The formulas for prices of 0-coupon bonds and one year forward rates are presented in Table 18.3 and (18.9). These formulas are repeated in (19.2) for convenience. The update of Table 19.1, with two extra rows, may be found in Table 19.3. This table will be used throughout this chapter.

(**19.2**) $$P_n = \frac{1}{(1+r_n)^n}; \qquad 1 + f_{i-1,i} = \frac{P_{i-1}}{P_i}$$

Term to maturity	1	2	3	4	5
Spot rate, r_n	2%	3%	3.5%	3.75%	3.8%
Price of *n* year 0-coupon bond of 1, P_n $=v(n)$; also the discount factor	0.980392	0.942596	0.901943	0.863073	0.829876
One year forward rate for year *n*, $f_{n-1,n}$	2%	4.0098%	4.5073%	4.5037%	4.0002%

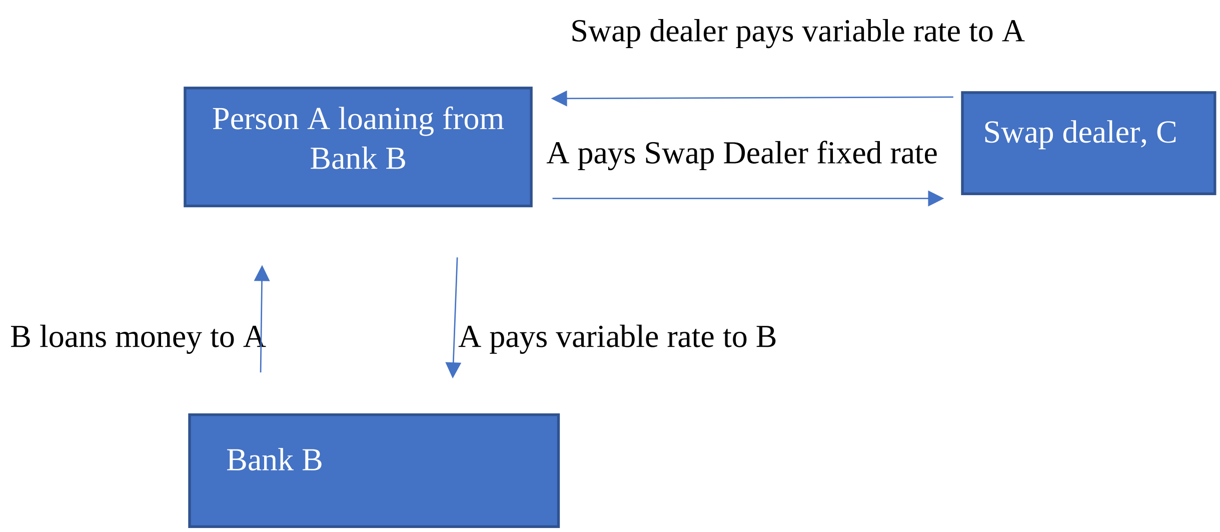
Table 19.3: *Table 19.1 augmented with rows for the price of 0-coupon n-year bonds and one year forward rates.*

19.2 Swaps: Suppose person *A* owes Bank *B* interest on a 100,000 3-year loan at a variable interest rate of the current spot rates. Then *A* will pay the amounts given in the *A* row of Table 19.4. Notice that interest rates are always on the loan amount, also called the *notional amount*, with no deduction of principal. This will be the approach when dealing with swaps. We have not yet explained the bottom row but will do so after some more background.

Time	0	1	2	3
A is obligated to pay *B*		100000*2%=2000	100000*4.0098%= 4009.80	100000*4.5073%= 4507.29
Fixed actuarially equivalent payments made by *A* to a 3rd party *C*		100000*0.034711 = 3471.14	100000*0.034711 = 3471.14	100000*0.034711 = 3471.14
Net amount that *A* pays *C*. Negative amounts indicate that *C* pays *A*.		-1471.10	538.70	1036.19

Table 19.4: *Interest payments from person A to 3rd party C.*

Recall that variable rates are tied to the *prime rate* or *LIBOR*. The prime rate or LIBOR gives rise to the term structure of Table 19.3. Table 19.4 is calculated based on values today of spot rates. But if the Prime rate or LIBOR changes next year then *A's* obligations will change? Why? Because the contract between *A* and *B* states that the interest payment each year will be based on the *prime rate* and LIBOR at *that time*. Those rates may create a new term structure. Bank *B* never committed itself to fixed numerical values in its contract. *If the prime rate or LIBOR rate goes up, the bank charges A more money; A wishes to avoid paying more money and seeks to protect himself. In finance, protecting oneself is called* hedging. *You hedge with a swap and a main purpose of swaps is to avoid the uncertainty of payments. How?* See the Figure below which we now explain.



A enters a swap deal with a swap dealer. The swap dealer typically enters into many deals. Sometimes people want the fixed rate and sometimes the variable rate. Let us just look at A's deal.

- A is obligated to pay Bank B the variable rate which may go up causing A a loss
- Instead A pays the swap dealer a fixed rate (So A can't lose money from changing rates)
- The Swap dealer pays A the variable rate
- A takes the variable rate received from the swap dealer and pays it to the bank
- The bank gets the variable rate it wants but A does not lose money from changing rates.
- We say that A has *hedged* (protected) his uncertainty about changing variable rates with a swap.

But how does the swap dealer know the fair fixed rate to charge A? This is done using actuarially equivalent payments and is illustrated in Table 19.5. The swap is computed at time 0, now and is an actuarially equivalent series of interest payments at some fixed rate, *R*. We can understand this as an equation of timelines as shown in Table 19.5. Note that the discount factors, v(t) from Table 19.3, in force today are used.

Time	0	1	2	3		0	1	2	3
Cashflow		100000*$f_{0,1}$	100000*$f_{1,2}$	100000*$f_{2,3}$			100000*R	100000* R	100000*R
Discount Factor, v(t)		$v(1)$	$v(2)$	$v(3)$			$v(1)$	$v(2)$	$v(3)$

Table 19.5: *The left timeline represents obligations of A. The right timeline represents fix-interest rate obligations. Typically A would enter the swap with a 3rd party, C. A pays the fix amounts and C pays the variable amounts.*

The EOV for Table 19.5 is presented in (19.6)

$$(\mathbf{19.6}) \qquad 100000 f_{0,1} v(1) + 100000 f_{1,2} v(2) + 100000 f_{2,3} v(3) = 100000 R v(1) + 100000 R v(2) + 100000 R v(3) \longrightarrow R = 3471.14$$

Remember, the forward rates and discount factors (the price of *n*-year zero coupons bonds maturing for 1) are listed in Table 19.3 So (19.6) is a linear equation in *R* and can easily be solved for. The resulting payments are shown in Table 19.4.

Let us review what happens

- *A,* the *payer* pays *C* the fixed amount of 3471.14.
- *C,* the *receiver* pays *A* (who pays *B*), the variable amounts of 2000, 4009.80, 4507.29
- Note that *A's* amount is fixed while *C* is taking risk that the variable rates will remain the same
- In practice, payments are *netted*. So *A* pays *C* the netted amount exhibited in 19.4.

19.3 Swap Terminology: Many terms have been introduced. These terms as well as other terms connected with swaps are compactly summarized in Tables 19.6a, 19.6b, 19.6c.

Term	Swap	Payer	Receiver	Counterparties	Netted payment
Meaning	Actuarially equivalent exchange of a variable rate obligation for a fixed rate obligation	The person who pays the fixed rate	The person who pays the variable rate and receives the swap rate	Payer and receiver are each counterparties	The difference between the fixed amount and variable amount
Example	Table 19.4 and (19.6)	*A*	*C*	*A*, *C*	See table 19.4 for an example

Table 19.6a: *A variety of terms connected with swaps.*

Term	Settlement period	Term, Tenor	Settlement dates
Meaning	The typically equally spaced periods at which payments are made	The number of periods of payment	The dates on which payments are made
Example	In our example a year	3	

Table 19.6b: *A variety of terms connected with swaps*.

Term	Notional Amount	Amortizing swap	Accreting
Meaning	The amount by which forward or fixed rates are multiplied. It is usually fixed.	Notional amount decreases over time	Notional amount increases over time
Example	100,000		

Table 19.6c: *A variety of terms connected with swaps.*

Section 19.4 A Simpler Formula: There is an alternate formula to (19.6), much easier computationally, to solve the equivalence of timelines presented in Table (19.5). Suppose I deposit 1 at time 0. At each time t the bank will give me R interest. At time n I withdraw the 1. My deposits and withdrawals are actuarially equivalent to the bank's deposits. This is shown in Table 19.7 and (19.8)

Time	0	1	2	3…	n
My Cashflows	1				-1
Banks cashflows		R	R	R…	R

Discount factors		*v(1)*	*v(2)*	*v(3)…*	*v(n)*

Table 19.7: *Cashflows of me and a bank which are actuarially equivalent*.

(**19.8**) $1 - P_n = RP_1 + RP_2 + \ldots + RP_n$

Table 19.9 and (19.10) shows the same set up but with variable forward rates.

Time	0	1	2	3…	*N*
My Cashflows	1				-1
Banks cashflows		$f_{0,1}$	$f_{1,2}$	$f_{2,3}$….	$f_{n-1,n}$
Discount factors from Table 19.3, $v(n)=P_n$		*v(1)*	*v(2)*	*v(3)…*	*v(n)*

Table 19.9: *Cashflows of an investor and a bank with variable rates*.

(**19.10**) $1 - v(n) = f_{0,1}v(1) + f_{1,2}v(2) + \ldots + f_{n-1,n}v(n)$

We can combine (19.6), (19.8) and (19.10) to obtain the following elegant (easier to compute) formula for computing:

(**19.11**) $f_{0,1}P_1 + f_{1,2}P_2 + \ldots + f_{n-1,n}P_n = 1 - P_n = RP_1 + RP_2 + \ldots + RP_n \longrightarrow R = \dfrac{1 - P_n}{P_1 + P_2 + \ldots + P_n}$

Applying this to the situation in Table 19.4 we obtain

$$R = \frac{1 - P_3}{P_1 + P_2 + P_3} = \frac{1 - 0.901943}{0.980392 + 0.942596 + 0.901943} = 0.0347114$$

Notice how this is much simpler than solving (19.6). Also notice that (19.11) does not require computation of the *forward rates;* it suffices to use the *prices* of zero coupon bonds maturing at time *n*.

Section 19.5 Deferred Swaps: Suppose a five year loan is made with the first two years at a fixed rate and with the remaining years at a variable rate. The lender may wish a 2-year deferred swap to deal with the last three years. Here, the deferral of two years means that the 0 point is set at time t=2. Notice that although the 0 point is deferred 2 years, the *first* payment is at time $t=3$. Table 19.5 now becomes Table 19.12 while (19.6) becomes (19.13). Finally, (19.11) becomes (19.14).

Time	2	3	4	5		2	3	4	5
	0	1	2	3		0	1	2	3
Cashflow		$100000*f_{2,3}$	$100000*f_{3,4}$	$100000*f_{4,5}$			$100000*R$	$100000*R$	$100000*R$
Discount Factor to 0		P_3	P_4	P_5			P_3	P_4	P_5

Table 19.12: *Timelines for a deferred swap. The left side is for the variable rate; the right side is for the fixed rate.*

(**19.13**) $$100000f_{2,3}P_3+100000f_{3,4}P_4+100000f_{4,5}P_5+ = 100000RP_3+100000RP_4+100000RP_5$$

(**19.14**) $$R = \frac{P_2 - P_5}{P_3 + P_4 + P_5} = \frac{0.942596 - 0.829876}{0.901943 + 0.863073 + 0.829876} = 0.043439$$

Just to recap: At time t=0, we calculate the fixed swap rate for a 3-year, 2-year deferred swap as 0.0434. This calculation is:

- Based on spot rates (prime rates and LIBOR rates) at time 0
- An exchange based on time 2
- Has the first payment(s) at time 3.

Section 19.6 Non Level Notional Values: A simple inspection of (19.6) shows that the notional value cancels from both sides of the equation. This gives rise to an important point: *If the notional value if fixed, which it usually is, the EOV can be written without it!*

However, if the notional value is non-level, the EOV must contain the notional values. Furthermore, there is no equivalent of the simple shortcuts of (19.11) and (19.14) for non-notional values. We have already indicated, in such a case, that if you are given (19.1) you should convert it to (19.2).

Illustrative Example: A 2-year deferred, 3-year amortization swap uses decreasing notional values of 300,000, 200,000 and 100,000. Compute the swap rate.

Illustrative Solution: Table 19.12 becomes Table 19.14

Time	2	3	4	5		2	3	4	5
	0	1	2	3		0	1	2	3
Cashflow		$300000*f_{2,3}$	$200000*f_{3,4}$	$100000*f_{4,5}$			$300000*R$	$200000*R$	$100000*R$
Discount Factor, $v(t)=P_t$		P_3	P_4	P_5			P_3	P_4	P_5

Table 19.14: *Timelines for a deferred swap with non-level notional values. Technically, we should discount to 2. But if two quantities are actuarially equivalent at t=2, they are also equivalent at t=0 (the original 0). This makes the computation easier.*

Equation (19.13) becomes (19.15)

(**19.15**) $300000 f_{2,3} P_3 + 200000 f_{3,4} P_4 + 100000 f_{4,5} P_5 = 300000 R P_3 + 200000 R P_4 + 100000 R P_5 \longrightarrow R = 0.044261$

Notice, that although (19.15) is cumbersome, it is still linear and can easily be solved, especially if Table 19.1 has been filled into Table 19.3. However, we have only computed the swap *rate*. The netted *payments* for each year must be computed separately using the notional amounts.

- For year 3, 300,000*(0.045073 – 0.044261) = 243.52
- For year 4, 200,000*(0.045036-0.044261) = 155.02
- For year 5, 100,000*(0.40002 – 0.044261) =-425.88

It is straightforward to calculate the absolute value of each netted payments: Simply subtract the smaller number (for example, for the first bullet, 13,278=300,000*0.044261) from the larger number (e.g., 13,522= 300000*0.045073). The *hard part* is figuring out who gains money and who loses money. So my advice is first to compute the absolute value of the netted payment and then to ask the following questions (We use the first bullet to illustrate):

- Which is bigger, the fixed rate, 0.044261 or the floating rate, 0.045073?
- In this case, the floating rate 0.045073 is bigger than the fixed rate 0.044261.
- So the person paying the bigger amount, the person paying the floating amount, loses money because (s)he pays more
- Contrastively, the person paying the smaller amount, the person paying the fixed amount, gains or receive the money because (s)he pays less
- The rules for the netted payment is that there is one net payment from the person losing the money (in this case the person paying the floating amount) to the person gaining the money (in this case the person paying the fixed amount).
- If netting is allowed, the actual amounts of $13,278 and $13,522 are not however exchanged.

Section 19.7 Market Value: A swap is like any other financial instrument. It can be sold. Therefore, it is necessary to sometimes price it at times other than 0. Let us consider the swap of Section 19.6. Suppose you are now at time t=3. You have just settled the payments for t=3. You wish to sell the remaining two payments. By t=3 the prime rate, LIBOR and spot rates *may* have changed. This is an important consideration. However, we will assume the implied forward rates are now the spot rates. Remember, however, the notional amounts and the fixed swap rate remain the same. The discount factors also change to be discounts from times t=4,5 to t=3. The pricing of the remaining two payments of the swap now becomes straightforward and are shown in Table 19.13. The price is simply the netted aggregate payment, the difference in payment between the payer and receiver.

Time	3	4	5		3	4	5
	0	1	2		0	1	2
Cashflow		$200000 * f_{3,4}$	$100000 * f_{4,5}$			$200000 * R$	$100000 * R$
Discount Factor, $v(t)=P_t$		P_4	P_5			P_4	P_5

Table 19.13: *Timelines for a deferred swap with non-level notional values.*

We have the following EOV (Notice the -243.52v(3): The amount is computed at t=3 and we have discounted all items to t=0 so that the amount at t=3, 243.52 must be discounted to 0 using the discount factor v(3). The numerical computation is achieved by dividing the left hand side after plugging in by v(3). All numbers are as in Table 19.3 and equation (19.6).

$$\textbf{EOV}: \quad \left(200000 f_{3,4} v(4) + 100000 f_{4,5} v(5)\right) - \left(200000 R v(4) + 100000 R v(5)\right) = -243.52 v(3)$$

This pricing of the remainder of the swap is negative for the payer and positive for the receiver.

Section 19.8 Net Interest Rate: There are times when a swap is coupled with a loan. The payer, in addition to the netted payment, must make interest payments. The total payments, the sum of the netted payment and loan interest payment, is called the *net interest amount.* We give an example based on the other examples in this chapter.

We have formulated everything thus far in terms of spot rates. Recall however, that the driving force of spot rates is the LIBOR and Prime Rate.

- Let us therefore suppose that the spot rates in Table 19.3 are the Prime rate + 25 bps.
- Let us further suppose that Table 19.4 represents a swap between payer *A*, who pays the fixed rate, and receiver *C*, who pays the floating rate.
- Now let us additionally suppose that *A* takes a loan of 100,000 from *C* and agrees to pay *C* the prime rate +50 bps.

Under these conditions we calculate the net interest payment from A to C in year 2.

- The calculations of the swap rate, *R*, in Table 19.4, as well as the payments remain the same (The loan does not change them).
- In year 2, the payer, *A*, makes a payment based on the fix rate to the receiver. The payment is shown in Table 19.4, 100000*0.034711 = 3471.14.
- In year 2, the receiver, *C*, make a payment based on the variable rate to the payer. The payment is shown in Table 19.4, 100000*4.0098%=4009.80.
- So A pays 3471.14 and receives 4009.80,
- So the netted swap payment from A to C is 3471.14 - 4009.80 = -538.66
- But *A* has loaned 100,000 from *C* and agrees to pay the Prime rate + 50bps. Since the spot rate is Prime rate + 25bps, it follows that *A* pays the spot rate + 25bps. Thus *A* will make an additional payment of 100,000*(4.0098%+0.25%) = 4259.80.
- It follows that the net payment from *A* to *C* is 4259.80-538.66= 3720.14.
- An alternative derivation is to add the 3471.14 fixed payment that A makes to the swap dealer plus the 25 BP x 100,000 = 3471.14+250=3721.84 (The variable rate is paid to the bank via the swap dealer; A merely receives the variable payment and transfers it so it is not part of A’s payment).

Source of Problems and Solutions: The SOA handout on Determinants of Interest Rates, Section 8, has 13 problems with worked out solutions. I see no reason to use the QIT problems as these problems are superior.

https://www.soa.org/Files/Edu/2016/edu-2016-fm-25-17-interest-rate-swaps.pdf

CHAPTER 20

Determinants of Interest

20.1 Borrower – Lender Equilibrium: Interest is the price at which money value is rented. In any transaction we have two parties, the *borrower* and *lender*. The lender is willing to have his money tied up by the borrower but only if the return, expressed as an interest rate, is sufficient. For example, the lender would not lend at a low rate if the lender thought (s)he could make more.

Similarly, the lender will only borrow money for a certain period if the rental value of the money is sufficiently low. The lender for example, would not pay a high rate of interest to start a business unless (s)he thought (s)he could make at least that much.

At any point of time each potential borrower and each potential lender will have thresholds for interest. The *equilibrium* rate represents that rate that is acceptable to *both* the borrower and lender. The equilibrium rate is one determinant of interest rate.

20.2 Methods of Rate Quotation: Interest rates will depend on how *secure* the lender is perceived. The U.S. government, or the Canadian government are perceived as good lenders. Neither of them have ever defaulted on a loan. The interest rate that they give on loans can be thought of as a rate without risk of *default.* A zero coupon bond issued by the U.S. or Canada is called a *treasury bill,* or *T-bill,* if its maturity is less than a year. The various methods of quoting T-bill rates along with the formulas for computing them are presented in Table 20.1

Method of rate quotation	**Effective Rate, i_E**	**Annual continuous rate, i_C**	**U.S. T-Bill convention, i_{US}**	**Canadian T-Bill convention, i_{CAN}**
Algebraic Formula	$i_E = (M/P)^{365/D}-1$	$i_C = \ln(1+ i_E)$	$i_{US} = 360/D \times I/M$	$i_{CAN} = 365/D \times I/P$
Numerical Formula	$(10000/9900)^{365/90}-1$	ln(1.041602)	360/90 x 100/10000	365/90 x 100/9900
Numerical example with formula	4.1602%	ln(1.041602)=4.0760%	4.0000%	4.0965%

Table 20.1: *4 methods of rate quotation for a D=90-day T-Bill maturing for M=10000 with price of P=9900. Note that I=M-P = 10000-9900=100.*

Typical Questions: The following are typical types of questions based on the above table:

- Given any 3 of *D, I, M, P* compute the fourth and compute the difference between two of the above rates

- Given the U.S. and Canadian citations and one of *M* or *P* calculate the effective and continuous rate

20.3 Four Explanations of the Increasing Yield Curve: Other things being equal, one expects higher effective rates of return from an instrument that takes longer to mature. For example, a 2 year bond might yield 1% while a 4 year bond might yield 1.5%. Why? There are 4 theories to explain the typically increasing nature of the yield curve. These theories are summarized in Table 20.2.

20.4 Four Types of Yield Curves: The yield curve is the graph corresponding to the collection of points (t, i_t). Typically, interest rates increase. Thus the typical yield curve has an increasing slope. It is therefore called an increasing yield curve or a normal curve. There are 3 other types of curves. They are summarized in Table 20.3

Name of Theory	Brief Explanation
Market Segmentation Theory	The group of people willing to borrow or lend for 4 years is much different than the group of people willing to borrow or lend for 2 years. Hence the equilibrium point at which each group will lend is different. For example, people willing to lend for 4 years are typically saving for a long term goal like retirement while people willing to lend for 2 years are typically saving for a short term goal
Liquidity preference theory or Opportunity cost theory	People prefer not to tie their money up for too long since if they do they might miss out on a good investment opportunity that arises. Hence people will charge higher interest rates for tying up their money for longer periods. (Note: The ability to *immediately* obtain funds is called *liquidity*)
Expectation theory	Long term interest rates give information on expected future short term rates. For example, if the 2 year rate is 1% and the four year rate is 1.5% then the implied forward one year rate at times 2 and 3 will be greater than 1%.
Preferred habitat theory	Like the market segmentation theory, the preferred habitat theory assumes that different groups of people have different equilibrium points. However, the preferred habitat theory also assumes that given enough money (higher rates) people will switch their preferred equilibrium points.

Table 20.2: *Four theories explaining why different maturity dates typically have different yields*.

Yield curve type	Normal, increasing	Flat	Decreasing	Bow shaped
Description	Slope is positive	Slope is 0 (all yields are the same)	Slope is negative	Slope is first increasing then decreasing

Table 20.3 *Four types of yield curves. Each curve type has a different shape.*

20.5 Default: Default refers to the inability to meet an obligation. For example, suppose I loan 1000 and agree to pay it back in 2 years. If two years from now I can't pay anything, I have *defaulted* on the loan. If I can only pay 250, I have *defaulted* on the loan but can make a partial payment.

In this section, we study how defaults naturally raise interest rates. Consequently, defaults are one determinant of interest rates. For purposes of this section we imagine that I have one million dollars and intend to make 1000 one-year loans of 1000 each. How much should I ask to be repaid? Let us call the repay amount *X*. Table 20.4 presents computation of rates for four scenarios of default:

Table 20.4 shows how default is one determinant of interest: If I expect a certain number of defaults with or without certain partial recoveries than to break even I must charge a higher rate.

Section 20.6 Inflation: This was discussed in Chapter 6. Suppose *goods* are inflating cost at 50% per annum. So, if 1 is needed to purchase milk/ oj / beer today, we need 1.50 next year. That means if I loan 1 at time t=0, I should charge 1.50 at t=1, in other words, a 50% interest rate. Thus we immediately see that inflation is one determinant of interest rates.

The above analysis holds in a world with perfect knowledge of inflation of both goods and wages. However, if the amount of inflation is unknown, or, as frequently happens, if some goods inflate at one rate and other goods at another rate, and wages inflate at different rate, then the differences generate uncertainty. A lender would still want an increased rate to compensate for the increase in the expected cost of goods. However, borrowers may have different assessments of the inflation. It is then not clear what the equilibrium point is between lenders and borrowers under inflation.

Assumption on defaults	**EOV, Equation to be solved for return of million (X=repayment amount after considering default)**	**Repayment amount**	**Interest Rate Factor (Credit spread due to default)**
0 defaults	1000 repayments * X = 1,000,000	X=1000	1000/1000 = 1.00 so credit spread is 0%
• **3 defaults per 1000,** • **no partial repayment**	997 repayment * X =1,000,000	X=1,000,000/997=1003	1003/1000=1.003; so credit spread is 0.3%
• **3 defaults per 1000;** • **25% partial repayments**	997*X + 3*25%X = 1000000	X=1,000,000/997.75=1002.02	1002.02/1000=1.002; so credit spread is 0.2%
• **6 defaults per 1000;** • **50% partial repayment;** • **4% is expected return (annually)**	994X + 6*50% X = 1,000,000 x 1.04	X=1,000,000*1.04/997=1043.13	1043.13/1040=1.0029 so credit spread is 0.29% (Note: ln(1.0029)=.0029 so continuous=exact here)

prior to considering defaults			

Table 20.4: *Four scenarios of default – no defaults, 3 complete defaults, 3 defaults with 25% repayment and 6 defaults with 50% repayment with an initial desire for a 4% return before considering defaults. For the first 3 cases, the interest rate factors needed to break even are given; for the 4th case the interest rate factor to obtain the desired yield is given. The corresponding credit-spread is given in all four cases.*

One measure of inflation is determined by the Consumer Price Index, or CPI. The government examines a "basket" of goods including groceries, utilities, rent etc. for a "typical" size American family. It can then estimate the cost of this basket in different geographic localities. The published CPIs can indicate pricing differences between states as well as reasonable estimates of inflation.

There are two types of loans:

- Those with inflation protection
- Those without inflation protection

Let us look at a simple example of an inflation protection bond:

Illustrative Example:

- I make a loan of 1000 for two years
- The reference amount I will get if there is no actual inflation is based on an annual rate of -.1001% This means that if there is no inflation I agree to receive back in two years for my loan of 1000, 1000 x (1-0.1001%) x (1-0.1001%) = 998. Why should I loan 1000 and agree to get back 998? Because I only get back 998 if there is no inflation. If I expect inflation I am paying 2 (2=1000-998) as the *cost for inflation protection*.
- At time t=2, I can look at say the CPI and determine that the rates for the inflation (of goods) rates for the past 2 years are 1.5% and 2.5%

Determine how much I get back at time t=2.

Illustrative Solution:

- My reference rate (if there is no inflation) is 1000 x (1-0.1001%) x (1-0.1001%) = 998
- My compensation at t=2 is 998 x 1.015 x 1.025 = 1038.29.

Illustrative Example and Solution continued: Suppose the CPI reported 1% inflation for the past 2 years. What am I compensated.

Ans: I am compensated 998 x 1.01 x 1.01 = 1018.06

As this example shows, my compensation depends on the actual inflation rate as well as the cost of inflation.

Section 2.7 Determinants of Interest: In previous sections we have seen a variety of determinants of interest such as s, the determinant of interest due to default as well as r, the determinant of interest due to deferral of compensation (The longer the loan term the higher the interest rate). How are these determinants combined. There are two methods?

- Factor method: If the rates are discrete effective then $1+i = (1+r)(1+s)$
- Addition method: If rates are continuous (so $c^{\text{continuous rate}} = (1+ \textit{continuous rate})$), then we are multiplying factors of the form $e^{\text{rate} \times t}$ so that rates are added: $i = r+s$. $e^i = e^r e^s$.

Throughout the rest of the section we will use the addition method because it is expositionally simpler. However, if you are given effective rates, use of the factor method is not that difficult.

Table 20.5 summarizes the components of interest rates for inflation-protected loans and non-inflation loans. The factors as well as the actual interest rate are indicated along with comments on the factors.

Section 20.8: Types of Lenders and Borrowers: In this section, we discuss various institutions that do lending and borrowing. The entire section is descriptive at a very high-level. The following institutions are covered

I. Banks
II. Governments & Large Corporations
III. States
IV. Canadian Treasury Bonds
V. Corporate bonds
VI. Central Banks
VII. The Federal Reserve system

I: Banks (including Savings and Loan associations) are intermediaries between lenders and borrowers.

Banks:

- Receive from depositors in the public
- Loan to individuals, small businesses, corporations

For a bank to stay in business:

- The interest rates charged on loans must be greater than the interest rates paid on deposits;
- The difference between these rates must be sufficient to:
    - Cover overhead costs and the losses on loans that go into default, as well as
    - Provide a reasonable amount of profit for owners.

Alternatives to banks:

- Alternative lenders (Communicate with investors to raise cash; but not regulated)
- Fintech companies (e.g. PayPal, apple pay, bitcoin)

Banks issue the following:

- Savings accounts, (you can withdraw your money at any time; you obtain interest)
- Checking accounts (you may not get interest or get lower interest rates than savings)
- CD (you cannot withdraw your money without penalty but get higher interest rates)

Factor	Explanation
R	Interest rate component for deferral. Recall that borrowers are willing to pay *more* for a loan of longer duration and lenders ask more for a loan of longer duration (See Table 20.2)
S	Interest rate component of default (See Table 20.4). *s* is also called the *credit spread* or the *spread for credit risk*
i_e	The expected rate of inflation.
i_u	Unexpected inflation. Note, if the economy is stable it will be *growing*, that is, the net worth of goods and services this year will be greater than those available last year. This increase represents unexpected inflation.
C	The cost of inflation, used in inflation protected bonds (See the illustrative example in Section 20.6)
Rate	Equation
Real interest rate for loans with inflation protection without consideration of default	$R_1 = r - c$
Actual interest rate on a loan with inflation without consideration of default	$r - c + i_u$
Nominal interest rate for loans without inflation protection without risk of default	$R_2 = r + i_e + i_u$
Interest rate without inflation when defaults are possible	$R^* = r + s + i_e + i_u$
Observable	
r, c, i_e, i_u	Cannot be observed individually directly
$R_2 - R_1 = i_e + i_u + c$	Can be observed directly. Overestimates the market's expectation of inflation. However, if $i_u + c$ is small relative to i_e, this does estimate expected inflation.

$R^* - R_2 = s$	Used in practice to estimate the spread of credit risk
$R_1 < 0$	Possible. This indicates that the cost of inflation protection is greater than the compensation for deferred compensation
$R_2 < 0$	Possible. Since i_u>0 always, R_2<0 usually means i_e<0 which means we expect prices to *fall*.

Table 20.5: *Comprehensive list of i) determinants of interest, ii) various interest measures with and without inflation protection and with and without consideration of default, iii) what is observable and not observable.*

Interest rates from Banks on savings depends on:

- # Investors (Higher for large #)
- Growth appetite (Higher rates if the Bank wishes to grow)
- Credit rating of bank

Lending activities of banks include

- Credit cards
- Personal loans
- Auto loans
- Mortgages

Three categories of loans made by banks:

- Secured loans (If you fail on a mortgage bank can size the house as security)
- Unsecured loans (Can't size anything) such as credit cards
- Loans with guaranteed payments (3rd party such as federal government backs up payment)

Borrower lending rate from a bank (Can be lower than posted rate) depends on

- Steadiness of pay check (lowers rate)
- Credit history of timely payments (lowers rate)

II: Governments and large corporations need too much money to obtain from banks. So, they raise money the following ways:

- Bonds
  - E.g. Zero coupon bonds from Strips
  - US Government bonds (http://www.federalreserve.gov/releases/h15/ , https://research.stlouisfed.org/fred2/)

    - Nominal return bonds (Risk free rate)
    - Real return/Treasury Inflation Protected Bonds (TIPS) (Good for pension funds)
- Stocks (if corporation is non-government, for-profit)

III: State Government Bonds (e.g. finance construction of water systems, roads, bridges) have the following attributes:

- Types
    - Revenue bonds (Revenue backed by collection authority e.g. for bridges)
    - General Obligation Bonds (backed by Taxes)
- Attributes: Rates can be higher or lower than Federal
    - Have a risk of default (NY, Detroit, California, PR(?))=> so rates higher
    - May be tax free (So a lower rate bond may give investor (e.g. retiree) more

IV: Canadian Treasury bonds issued by government of Canada have the following attributes:

- Can be in US dollars (to pay US obligations)
- Can be in Canadian dollars
- Like US T-bills (But maturities are 91 days to 30 years)
- Credit worthy (never defaulted)

V: Corporate Bonds, used to raise money for big corporations, have the following attributes:

- Can be Coupon bonds
- Ratings: e.g. AAA > AA >A > BBB.....with 1,m,h, as suffix - indicates probability of default
- Bid Price (What buyer wants) < Ask Price (What seller wants)
- Bid-Ask spread (Ask minus Bid price): higher means lower liquidity (ability to immediately sell)
- Call provision (Seller can redeem early); Put Provision (Buyer can redeem early)

VI: Central Banks are banks with the following attributes:

- Main function #1: Lender of last resort for banks in that country
- Main Function #2: Facilitate operation of country's payment system
- May be involved in more operations such as currency creation
- All banks must maintain a reserve with central bank (which is the source of central bank funds)

VII: US Central Bank = Federal Reserve Bank System

(http://www.federalreserve.gov/aboutthefed/structure-federal-reserve-banks.htm) has the following structure:

- Board of governors
- 12 regional banks
- Committee, FOMC, Federal Open Market Committee which does the following:
  - Sets federal funds rate (Rate at which banks lend their reserves to other banks)
  - Sets the Discount rate (Rate at which banks loan from the Federal Reserve bank)
  - Sets monetary policy = Sets federal funds rate

Source of Problems and Solutions: There are QIT problems on Determinants of interest, QIT#192,193,194,195. As of this version of the book, I have not posted solutions. The interested student however can study problems QIT#192,193,194,195, read the SOA solutions, and compare them to the approach in these notes.

CHAPTER 21

Callable Bonds

21.1 Callable Bonds: A callable bond is a bond that the *seller* can redeem at dates other than the maturity date. The contrast between a callable and not callable bond is summarized in Table 21.1.

Bond Type	**When can it be redeemed**	**Who can force redemption**
Ordinary Bond	Maturity date = n	Both buyer and seller
Callable Bond	Can be redeemed • At both time n as well as • At least one other time	• At n: Both buyer and seller • At other than n: Only seller

Table 21.1: Differences between callable and non-callable bonds

21.2 Types of Callable Bonds: Callable bonds are classified depending on when else, besides n, the bond can be redeemed. This is summarized in Table 21.2

Type of callable bond	**American option**	**Bermuda option**	**European option**
When can bond be redeemed	Any $t \leq n$	$t_1, t_2, t_3, \ldots, t_s, s \geq 1$	At $t=n$ and $t=t_1$
Comment	Any time prior to n	Only at finite number of dates; typically, at coupon dates. The Bermuda option is typically used in the course.	Only at one other date

Table 21.2: Three types of callable bond options

21.3 Yield of a Callable Bond: What is the yield of a callable bond? A little reflection shows that it depends when it is called, when it is redeemed. Table 21.3 shows the yields for a 10%, 3-year par bond, maturing at 1000, under various calling scenarios with a Bermuda option.

N	**I**	**PV**	**PMT**	**FV**	**Comments**
3	CPT=8.0578%	-1050	10%*1000=100	1000	Called at n
2	CPT=7.2259%	-1050	10%*1000=100	1000	Called at 2
1	CPT=4.7619%	-1050	10%*1000=100	1000	Called at 1
Comments		*Price fixed at time of purchase*	*Coupon amount, Fr, fixed at time of purchase*	*Redemption amount for each callable option fixed*	

				in contract In this case all redemption values are same but in other cases redemption values may change	

Table 21.3: The yield on a callable bond depending on when it is called.

So what is the yield? The *yield* is defined as the *least* yield or as the *minimum* yield over all possible redemption dates. As can be seen from Table 21.3, the yield to the investor of this callable bond is at least 4.7619%.

Section 21.4 3 Methods for Computing Minimum Yields of Callable Bonds: The following three theorems give three methods of computing the minimum yield

Theorem 21.1: The minimum yield of a callable bond may be obtained by creating a table similar to Table 21.3 listing all call dates. In computing the yield, the price and coupon amount remain the same but the redemption value may either change or remain the same.

Proof: Clear (Definition of minimum yield)

Theorem 21.2: Table 21.4 summarizes methods of computing minimum yield under certain conditions

Condition #1 (P,C)	**Condition #2 (C)**	**Condition #3 (Option)**	**Conclusion**
Bond bought at premium ($P > C$ or $g > i$ or (when C=F, $r > i$)	All redemption amounts are the same	Bermuda option – bond only redeemable at certain coupon dates	Minimum yield is on earliest call date
Bond bought at discount ($P < C$ or $g < i$ or (when C=F, $r < i$)	All redemption amounts are the same	Bermuda option – bond only redeemable at certain coupon dates	Minimum yield is on latest call date

Table 21.4: Methods of computing minimum yield if certain conditions are satisfied.

Illustrative Example: Consider Table 21.3.

- Condition #1: Bond bought at premium since P=1050 > 1000 = C
- Condition #2: All redemption amounts for t=1,2,3 are the same (1000)
- Condition #3: The callable type is Bermuda since bond can only be called on coupon dates
- Conclusion: It follows that the minimum yield is on the earliest call date, that is, the yield at t=1, 4.7619%, is the minimum yield.

The point of the theorem is that Theorem 21.1 need not be used.

The proof of Theorem 21.2 is presented in Section 21.5.

Notice that Theorem 21.2 might be complicated to apply; it is easy to confuse *premium* and *discount* and easy to confuse earliest vs. latest. The following result, due to me, is simple to apply.

Theorem 21.3: Table 21.5 summarizes methods of computing minimum yield under certain conditions

Condition #2 (C)	Condition #3 (Option)	Conclusion
All redemption amounts are the same	Bermuda option – bond only redeemable at certain coupon dates	Minimum yield is the smallest of the earliest and latest yield. No intervening yields need be computed

Table 21.5: Simplified version of Theorem 21.2

Illustrative Example: Consider Table 21.3.

- Condition #1: All redemption amounts for t=1,2,3 are the same (1000)
- Condition #2: The callable type is Bermuda since bond can only be called on coupon dates
- Conclusion: It follows that the minimum yield is either on the earliest call date, t=1, 4.7619%, or on the latest call date, t=3, 8.0578%. Since 4.7619% < 8.0578%, it follows that the minimum yield occurs at t=1, at a yield of 4.7619%. Note: We do not have to evaluate the yield at t=2 since we are certain it will be between these two yields.

Proof: Theorem 21.3 follows immediately from Theorem 21.2.

Section 21.5 Proof of Theorem 21.2: We need the following result.

Lemma 21.4: $a_{\overline{n}|i}$ is a strictly positive monotone decreasing function of i, equaling n at i=0 and going to 0 as i goes to infinity.

Proof: The values of the annuity at i=0 and infinity are clear (say by l'hopitals rule). The strictly positive monotone decreasing nature of the function can be obtained by applying the quotient rule and differentiating with respect to the force of interest. The derivative is doable but computable. The derivative is always negative. This completes the proof.

Lemma 21.5: For a bond bought at a premium, the bond price as a function of interest rate is monotone decreasing.

Proof: Clear. Use $P = C(g - i)a_{\overline{n}|i} + C$

Proof that if bond bought for a premium ($g > i$) that minimal yield is on earliest date. Table 21.6 shows calculations of the bond redeemed at time t at yield y_t, minus the bond redeemed at time t+1, at the same price and same yield. In this case y_t=i. Since the bond is bought at a premium the difference is negative.

Time	0	1	2	3…	t	t+1
Redeemed at t for yield y_t	P_t	Cg	Cg	Cg…	$Cg+C$	
Redeemed at t for yield y_t	P_{t+1}	Cg	Cg	Cg	Cg	$Cg+C$

Difference of last 2 lines	0	*0*	*0*	*0*	*C*	$-C(g+1)$
Last line at t+1	$P_t - P_{t+1}$					$C(1+i)$-$C(g+1)$ = $C(i-g) < 0$

Table 21.6: Bond redeemed at time t at yield y_t minus bond redeemed at time t at yield y_t.

As can be seen P_t-P_{t+1} <0 or $P_t(i) < P_{t+1}(i)$ if both bonds have the same yield. But recall that with a callable bond the price remains the same. Since we must lower $P_{t+1}(i=y_t)$ to $P_t(i)$, Lemma 21.5 says we must raise the rate, that is with P_{t+1} replaced by P, $y_{t+1} > y_t$, that is the earlier yield is bigger. This completes the proof.

Section 21.6 Problems: Problems may easily be solved using the three theorems. Various variations are possible

- You can have several redemption values over many redemption dates. In such a case you apply the Theorems to each group of redemption dates by calculating one or two values. However, this must be done for each redemption value.
- The QIT problems show some very intricate problems where one has to price the bond at several dates and for each pricing find the minimum yield. Such problems involve decision trees and are typically very complicated and not illuminating (Though you should try at least one)
- Remember: You can always find the minimum yield by brute force, by considering all redemption dates and calculating yields.

Source of QIT problems: http://www.soa.org/Files/Edu/2017/exam-fm-sample-questions.pdf
Source of my solutions: www.Rashiyomi.com/math/
QIT Problems:

- QIT#54 Premium, $r > i$. Find C
- QIT#55 Price at several dates and find minimal yield for each price
- QIT#56 Price at several dates and find minimal yield for each price
- QIT#57 Discount, $P < C$, Find i
- QIT#91 Redemption possible at every other coupon date

SOURCE FOR ARCHIVED EXAMS: Exams are called *Course 2* or *FM* (Financial Mathematics)
https://www.soa.org/Education/Exam-Req/Syllabus-Study-Materials/edu-multiple-choice-exam-archives.aspx

- N05#22
- M05#11 Conceptual, hard